Test 1.

Maths Test

No calculators are allowed.

You can solve the problems in the order you like.

Exercice 1:

Give the number of divisors of 2010.

Exercice 2:

Show that $\sqrt[3]{5}$ is irrational.

Exercice 3:

Let x and y be two reals such that $0 < x \leq y \leq \frac{\pi}{2}$. Show that $\frac{\sin(y)}{\sin(x)} \leq \frac{y}{x}$.

Exercice 4:

Let consider a non-flattened triangle ABC. Let denote I the intersection point of the interior angle bissectors of $\angle B$ and $\angle C$ and J the intersection point of the exterior angle bissectors of $\angle B$ and $\angle C$.
We note I' the orthogonal projection of I on (BC) and J' the orthogonal projection of J on (BC).
Show that the center of segment $I'J'$ is the center point of segment BC.

Exercice 5:

Let X be a set with n elements.
Give the number of couples (A, B) with A and B subsets of X such that $A \subset B$.

Exercice 6:

We define $f_n(x) = x^n + x - 1$ with $n \in \mathbb{N}^*$.

a. Show that the equation $f_n(x) = 0$ has a unique root in $]0; 1[$. We denote x_n this root.

b. Show that the sequence $(x_n)_{n \geq 1}$ is increasing.

c. Is the sequence (x_n) convergent ? If yes, what is its limit?

Exercice 7:

A batch of people is submitted to a virologic test.
We assume that in case of contamination by the virus the test is positiv with probability 99%.
We also assume that in absence of contamination the test is positif with probability 5%.
The infection frequency is 0.01%

What is the probability that an individual having a positiv test is effectively infected?

Exercice 8:

Find the antiderivative of $\dfrac{x^3}{1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}}$ over $[0; +\infty[$.

Exercice 9:

We define $F_0 = 0$, $F_1 = 1$ and by induction the sequence $(F_n)_{n\geq 0}$ according to the recursive relation: $\forall\, n \geq 0,\ F_{n+2} = F_{n+1} + F_n$.

a. Show meticulously that for all integer $n \geq 0$, F_n is an integer.

b. Show that for all integer $k \geq 0$, $F_{3\,k}$ is even.

c. Show that if $m \in \mathbb{N}^*$ and $m \mid n$ then $F_m \mid F_n$.

Exercice 10:

Find all the integers x and y so that $x^3 - y^3 = 19$.

Exercice 11:

Let r be a real of $]0; \frac{\pi}{2}[$.

We define $f_n(t) = \frac{(r^2 - t^2)^n}{2^n\, n!}$ and $I_n = \int_0^r f_n(t) \cos(t)\, dt$.

Thus we have $f_0(t) = 1$, $f_1(t) = \frac{r^2 - t^2}{2}$.

a. Compute I_0 and I_1.

b. Show that, for $n \geq 2$, $f_n''(t) = -(2n-1)\, f_{n-1}(t) + r^2 f_{n-2}(t)$.

c. Deduce from it that $I_n = (2n-1)\, I_{n-1} - r^2 I_{n-2}$.

d. In this question, we assume that there exists an integer N such that for all $n \geq N$, $I_n = 0$.
Reach a contradiction using **a.** and **c.**

We assume for the following questions that r is a rational. We write $r = \frac{a}{b}$ with a et b natural integers.

e. Show that if $A > 0$, then $\frac{A^n}{n!} \to 0$. Deduce form it that $b^n I_n \to 0$.

f. Show that there are integers u_n and v_n such that $b^n I_n = u_n \cos(r) + v_n \sin(r)$.

g. We assume that $\tan(r)$ is rational and so that it can be written $\frac{p}{q}$, with p and q integers.

Justify that $b^n q \frac{I_n}{\cos(r)} \in \mathbb{Z}$.
Reach a contradiction.

h. Show that π is irrational.