Maths Lab: SMO 2009, Junior.

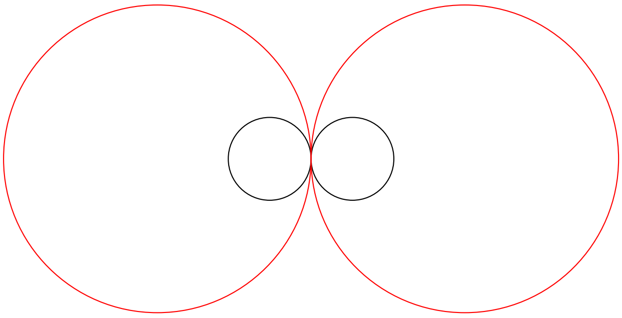
Maths Lab: Elements of solutions.

The ◊ indicates difficult questions.

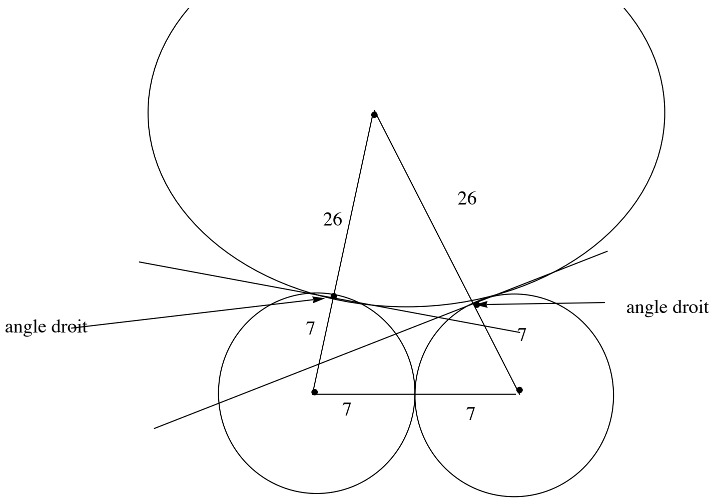
1

Answer (C).

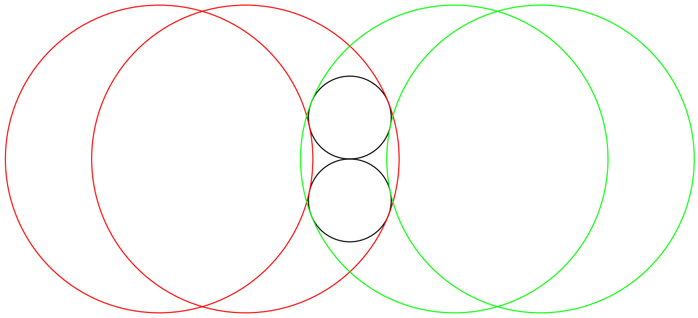
Supposing that the 3 ceters are aligned, we can built two circles tangent to the tangent points of the first two, with the centers aligned with the 2 first centers and the radius 26 gives only 2 possibilities.



Let's suppose that the three centers aren't aligned.



As Γ is tangent to C_1, it exists a point E on $C_1 \bigcap \Gamma$ and a line Δ such as $E \in \Delta$ and $\Delta \perp (O\,E)$ and $\Delta \perp (E\,\Omega)$.
Thus $(O\,E) \,//\, (E\,\Omega)$ and then $(O\,E) = (E\,\Omega)$.
So we deduce, $\Omega\,O = \Omega\,O' = 26 + 7$ or $26 - 7$, with O' the center of C_2 and then Ω belongs to the perpendicular bissector of $[O\,O']$.
We obtain 4 more circles.



2

Answer (A).

ABC and ODC are clearly isometric, so we can deduce that the shaded area is equal to the area between two quadrants, one of radius 4 and one of radius 8.

Then the area of the shaded area is given by $\frac{\pi}{4} \times 8^2 - \frac{\pi}{4} \times 4^2 = \pi(16-4) = 12\pi$.

3

Answer (D).

Method 1:
We can observe that $2 \le x \le 7$ as $\sqrt{\ }$ is defined on $[0; +\infty[$ and so we need $x-2>0$ and $7-x>0$.

We have $\left(\sqrt{x-2}+\sqrt{7-x}\right)^2 = 5 + 2\sqrt{(x-2)(7-x)} = 5 + 2\sqrt{6,25-(x-4,5)^2}$.

Hence the maximum value is reached for $x = 4,5$ because $6,25-(x-4,5)^2 \le 6,25$ for all x real.

Finally the maximum value is $k = \sqrt{4,5-2}+\sqrt{7-4,5} = 2\sqrt{2,5} = \sqrt{10}$.

Method 2:
Finding the derivative of $x \mapsto \sqrt{x-2}+\sqrt{7-x}$, we easily prove that the function is increasing on $[2; 4,5]$ and decreasing on $[4,5; 7]$.

So there is a maximum reached at 4,5 equal to $\sqrt{4,5-2}+\sqrt{7-4,5} = 2\sqrt{2,5} = \sqrt{10}$.

4

Answer (B).

Lets note that $\Delta\, BXY$ is isoceles so the midpoint I of $[XY]$ is also the foot of the altitude through the vertex B.
Thus $(BI) \perp (XY)$.
Lets note J the point of tangency between line (WZ) and the third circle: $(WZ) \perp (CJ)$.

Thus in the triangle WBJ, as $(BI) // (CJ)$ we have $\frac{BI}{CJ} = \frac{WB}{WC}$. We deduce $BI = 20 \times \frac{60}{100} = 12$.

Then in the right-angled triangle BIX, using the Pythagoreas Theorem, $BX^2 = BI^2 + IX^2$ so $IX^2 = 20^2 - 12^2 = 256 = 16^2$ and $IX = 16$.

As I is the midpoint of $[XY]$, $XY = 32$.

5

Answer (D).

We know that $y = \dfrac{10x}{10-x}$ so we can deduce that $y(10-x) = 10x$, $x(10+y) = 10y$ and thus $x = \dfrac{10y}{10+y}$.
As x and y are both negative integers, x and $10y$ also are negative integers, so $10 + y > 0$.
Therefore $-10 < y < 0$.

Then as x is an integer, $10 + y$ divide $10y$.
Lets try.
$y = -1$ gives $x = \dfrac{-10}{9}$: impossible, $y = -2$ gives $x = \dfrac{-20}{8} = \dfrac{-5}{4}$: impossible, $y = -3$ gives $x = \dfrac{-30}{7}$: impossible, $y = -4$ gives $x = \dfrac{-40}{6} = \dfrac{-20}{3}$: impossible, and finally $y = -5$, $x = \dfrac{-50}{5} = -10$.

6

Answer (C).

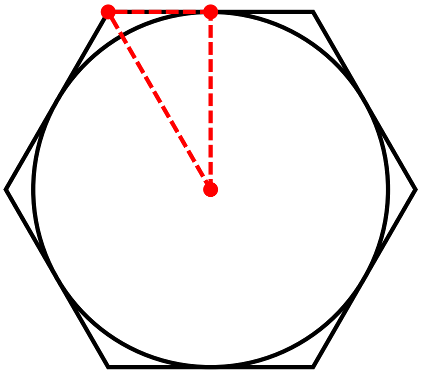
Note that $a_1 = 2009 + 1^2$, $a_2 = 2009 + 1^2 + 2^2$, $a_3 = 2009 + 1^2 + 2^2 + 3^2$, …, $a_{50} = 2009 + 1^2 + 2^2 + \ldots + 50^2$.

One must know that $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$, so $a_{50} = 2009 + \dfrac{50 \times 51 \times 101}{6} = 2009 + 25 \times 17 \times 101 = 44\,934$.

7

Answer (B).

Lets consider the circumscribded hexagon of the circle.



The plan is entirely covered by the hexagons, so the percentage of plane covered by the circles is equal to the percentage of 1 circle in his circumscribded hexagon.
As the length of the side of an hexagon is equal to the radius of his circumscribded circle, and as the circle we are intereseted in is tangent to each side of the hexagon in its midpoint, we deduce, using the Pythagoreas Theorem, that the side x of the hexagon is such that $x^2 = \dfrac{x^2}{4} + r^2$, where r is the radius of the circle.

So we have $x^2 = \frac{4}{3} r^2$ and $x = \frac{2}{\sqrt{3}} r$

Then the area of the hexagon is 12 times the area of the precedent right-angled triangle.
So the area of the hexagon is $12 \times \frac{1}{2} \times \frac{x}{2} \times r = 2\sqrt{3}\ r^2$.

The area of the circle is πr^2 so the percentage of the plan that is covered by the coins is $\frac{\pi r^2}{2\sqrt{3}\ r^2} \times 100 = \frac{50\pi}{\sqrt{3}}$.

8

Answer (E).

$|x + y + 1|^2 = (x + y + 1)^2 = x^2 + y^2 + 2xy + 2x + 2y + 1$.
So $|x + y + 1|^2 = 6 + 2\left(2 + 3\sqrt{2}\right) + 1 = 11 + 6\sqrt{2}$.

Thus $|x + y + 1| = \sqrt{11 + 6\sqrt{2}}$.

Lets note that $\left(a + b\sqrt{2}\right)^2 = (a^2 + 2b^2) + 2ab\sqrt{2}$.

Lets find 2 integers a and b such that $\left(a + b\sqrt{2}\right)^2 = 11 + 6\sqrt{2}$.
As we need $2ab = 6$, we deduce that either $a = 1$ and $b = 3$ either $a = 3$ and $b = 1$.
With $b = 3$, $a^2 + 2b^2 = 19 > 11$ and with $a = 3$ and $b = 1$, $a^2 + 2b^2 = 11$.

So $\sqrt{11 + 6\sqrt{3}} = \sqrt{\left(3 + \sqrt{2}\right)^2} = 3 + \sqrt{2}$.

Finally $|x + y + 1|^2 = 3 + \sqrt{2}$.

Remark:
One can try with the solutions given.

9

Answer (E).

$(x - 16)(x - 14)(x + 16)(x + 14) = \left(x^2 - 16^2\right)\left(x^2 - 14^2\right) = x^4 - 452x^2 + 50\,176 = X^2 - 452X + 50\,176$ with $X = x^2$.

We know that $X^2 - 452X + 50\,176$ reaches its minimum for $X = -\frac{-452}{2} = 226$.

So as $X = x^2$, with $x = \sqrt{226}$, we deduce that y reaches its minimum for $x = \sqrt{226}$ and then $y = 226^2 - 452 \times 226 + 50\,176 = -900$.

Remark:
Noticing that $x^4 - 452x^2 + 50\,176 = (x^2 - 226)^2 - 900$ is also a good way to find the minimum.

10 ◊

Answer (A).

We have $\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=1-\frac{1}{a}=\frac{a-1}{a}$, so $\frac{a}{b}+\frac{a}{c}+\frac{a}{d}=a-1$.

As $1 \le a < b < c < d$, $\frac{a}{b}+\frac{a}{c}+\frac{a}{d}<3$ so $0 \le a-1<3$ and then $1 \le a < 4$: $a = 1$, $a = 2$ or $a = 3$.

With $a = 1$, there is no solution.

With $a = 3$, $\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{2}{3}$, so $\frac{3b}{c}+\frac{3b}{d}=2b-3$ then $2b-3<6$, $b < 4, 5$ so $b = 4$.

Then $\frac{1}{c}+\frac{1}{d}=\frac{5}{12}$ and $\frac{12c}{d}=5c-12$ so $5c-12<1$, and $c < 2, 4$: impossible as $b < c$.

There is no solution with $a = 3$.

With $a = 2$, we obtain $\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=\frac{1}{2}$ so $\frac{1}{c}+\frac{1}{d}=\frac{b-2}{2b}$ and $\frac{2b}{c}+\frac{2b}{d}=b-2$.

Then $b-2<4$ so $b<6$ and so $a = 2 < b < 6$.

- With $b = 3$, we obtain $\frac{1}{c}+\frac{1}{d}=\frac{1}{6}$ so $\frac{6c}{d}=c-6$ and as $c<d$, it falls $c-6<6$.

  Then $3 < c < 12$.

  With $c = 4$, we obtain $\frac{1}{d}=\frac{1}{6}-\frac{1}{4}<0$, impossible.

  With $c = 5$, $\frac{1}{d}=\frac{1}{6}-\frac{1}{5}<0$, impossible.

  With $c = 6$, $\frac{1}{d}=0$!! Impossible.

  With $c = 7$, $\frac{1}{d}=\frac{1}{42}$ so (2; 3; 7; 42) is a solution.

  With $c = 8$, $d = 24$, with $c = 9$, $d = 18$, with $c = 10$, $d = 15$

  With $c = 11$ or 12, no solution.
- With $b = 4$, we find $c = 5$ and $d = 20$ or $c = 6$ and $d = 12$.
- With $b = 5$, no solution.

Finally there are 6 solutions.

11

Lets note x the length of the standard model.

Then $x^2+\left(\frac{4}{3}x\right)^2=20^2$ and so $25x^2=9\times 400$ and $x^2=144$ so $x = 12$.

The area of the standard model is then $12\times 16 = 192$.

Lets note y the length of the widescreen.

Then $y^2 + \left(\frac{16}{9}y\right)^2 = 20^2$ so $337\,y^2 = 81\times20^2$ and $y^2 = \frac{81\times20^2}{337}$.

The area of the widescreen model is $y\times\frac{16}{9}y = \frac{16}{9}y^2 = \frac{9\times16\times20^2}{337}$.

$\frac{A}{300} = \frac{192}{\frac{9\times16\times20^2}{337}} = \frac{337}{300}$ so $A = 337$.

Answer: $A = 337$.

12

Lets note P the area of the pentagon and R the area of the rectangle.
We have $B = \frac{3}{16}P$, so $A = \frac{13}{16}P$, and $B = \frac{2}{9}R$ so $C = \frac{7}{9}R$.

We deduce first that $\frac{P}{R} = \frac{\frac{16}{3}B}{\frac{9}{2}B} = \frac{16}{3}\times\frac{2}{9} = \frac{32}{27}$.

Second $\frac{A}{C} = \frac{\frac{13}{16}P}{\frac{7}{9}R} = \frac{13}{16}\times\frac{9}{7}\times\frac{P}{R} = \frac{13}{16}\times\frac{9}{7}\times\frac{16}{3}\times\frac{2}{9} = \frac{26}{21}$ so $\frac{m}{n} = \frac{26}{21}$ and then $m + n = 47$.

Answer: 47.

13

There are 2^9 different answer for 9 questions so we need $2^9 + 1 = 512 + 1 = 513$ scripts to guarantee at least two scripts with nine identical answers.

Answer: 513.

14

Six girls leave 7 spaces: …G…G…G…G…G…G….
There is in effet $6\times5\times4\times3\times2\times1 = 6!$ ways to arrange the 6 girls and $5!$ ways to arrange the 5 boys with each others.
We need to count where we put the 5 boys in the spaces left by the 6 girls.
For instance: BGGBGBGBGBGBG and so on.

It is then equivalent to count the number of words of length 7 we can make with 5G's and 2 blanks.
Assuming we can distinguish all of them, we obtain $7!$ possibilities, but as the 5 G's are identical and the 2 blanks also, we need to divide by 5! and 2! in order to avoid the repetitions.
Finally there are $\frac{7!}{5!\;\;2!} = \frac{1\times2\times3\times4\times5\times6\times7}{1\times2\times3\times4\times5\times1\times2} = 21$ possibilities.

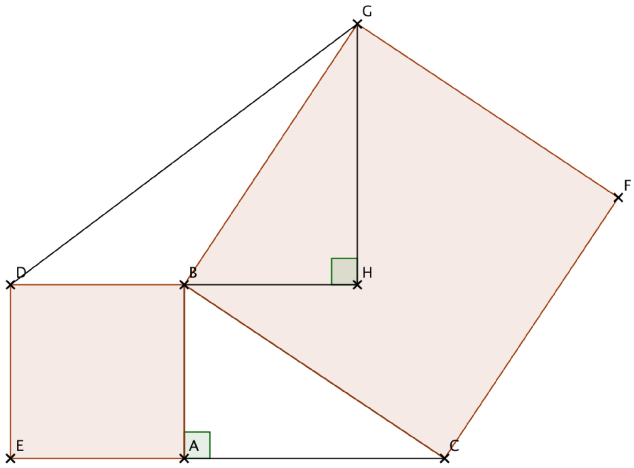
So the numbers of ways to arrange the 6 girls and the 5 boys is $21\times5!\times6!$ so $k = 21\times5 = 2520$.

Answer: 2520.

15

We can deduce the lengths of the sides of the right-angled triangle ABC : $AB = \sqrt{8} = 2\sqrt{2}$, $BC = \sqrt{26}$ and so $AC = \sqrt{18} = 3\sqrt{2}$.

As $BG = BC$ and $BD = AB$, we can build a right-angled triangle BHG as schown on the figure isometric to the triangle ABC.



The area of BHG is equal to the area of ABC and equal to $\frac{1}{2}\times 2\sqrt{2}\times 3\sqrt{2} = 6$.

Then as B is the midpoint of $[BH]$, the area of BDG is also equal to the area of BHG.

Answer: 6.

16

Remark:

For all positive integer k, $\frac{1}{n} - \frac{1}{n+k} = \frac{n+k-n}{n(n+k)} = \frac{k}{n(n+k)}$ so $\frac{1}{n(n+k)} = \frac{1}{k}\left(\frac{1}{n} - \frac{1}{n+k}\right)$.

We deduce then that $\frac{1}{n(n+1)(n+2)} = \frac{1}{n+1}\times\frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$.

The sum $\frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \ldots + \frac{1}{14\times15\times16} = \frac{1}{2}\left(\left(\frac{1}{2\times3} - \frac{1}{3\times4}\right) + \left(\frac{1}{3\times4} - \frac{1}{4\times5}\right) + \ldots + \left(\frac{1}{14\times15} - \frac{1}{15\times16}\right)\right)$ so $\frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \ldots + \frac{1}{14\times15\times16} = \frac{1}{2}\left(\frac{1}{2\times3} - \frac{1}{15\times16}\right) = \frac{13}{160}$.

Answer: 13 + 160 = 173.

17 ◊

Lets note that $a-b+2=a+1-(b-1)$ and $a\,b-a+b=(a+1)\,(b-1)-1$.

We have $a+\frac{1}{a+1}=b+\frac{1}{b-1}-2$ so $a+1+\frac{1}{a+1}-(b-1)-\frac{1}{b-1}=0$ and then $((a+1)-(b-1))+\left(\frac{1}{a+1}-\frac{1}{b-1}\right)=0$.

We obtain $((a+1)-(b-1))+\frac{(b-1)-(a+1)}{(a+1)\,(b-1)}=0$ so $((a+1)-(b-1))\left(1-\frac{1}{(a+1)\,(b-1)}\right)=0$.

As $a-b+2\neq 0$, $((a+1)-(b-1))\neq 0$ so $1-\frac{1}{(a+1)\,(b-1)}=0$ then $\frac{(a+1)\,(b-1)-1}{(a+1)\,(b-1)}=0$ and finally $(a+1)\,(b-1)-1=0$.

Thus $a\,b-a+b=2$.

Answer: 2.

18

Reminder:

$$|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}.$$

Assuming $y\geq 0$, we deduce $x=7\geq 0$ and then $7+7+5\,y=2$ so $y=\frac{-12}{5}<0$: absurd.

So $y<0$ thus $x-2\,y=7$.

Then assuming that $x\leq 0$, we deduce $5\,y=2$ so $y=\frac{2}{5}>0$: absurd.

Thus $x>0$.

Finally x and y are solutions of the system $\begin{cases} x-2\,y=7 \\ 2\,x+5\,y=2 \end{cases}$ which gives $\begin{cases} y=\frac{-4}{3} \\ x=\frac{13}{3} \end{cases}$.

Therefore $x+y+2009=\frac{13}{3}-\frac{4}{3}+2009=3+2009=2012$.

Answer: 2012.

19

Since the 2 sets are equal, their sums are equal too.
Thus $3\,p+3\,q=6\,n-27$ so $p+q=2\,n-9$.

$2\,n$ is even and 27 is odd so $2\,n-27$ is odd.

So $p+q$ is the sum of two consecutive prime numbers which is odd.
If $2<p$ and $2<q$, p and q are odds and then $p+q$ is even (the sum of two odds is even).
Thus p or q is even. The only even prime number is 2.
So $p=2$ and $q=3$.

We deduce $2\,n-9=5$ then $n=7$.

Answer: 7.

20

Lets note that:

$$\left(\sqrt{x}+\sqrt{y}+\sqrt{2009}\right)\left(\sqrt{xy}-\sqrt{2009}\right)=x\sqrt{y}+y\sqrt{y}+\sqrt{2009xy}-\sqrt{2009x}-\sqrt{2009y}-2009.$$

So as $\sqrt{x}+\sqrt{y}+\sqrt{2009}>0$, $\sqrt{xy}-\sqrt{2009}=0$ and $xy=2009$: x and y are divisors of 2009.

Or $2009=7^2\times 41$.

We have then the possibilities for (x; y) (ordered pairs) : (1; 2009), (7; 287), (41, 49), (49, 41), (287; 7) and (2009; 1).

Answer: 6

21

As $\frac{1}{2009}<\frac{1}{2003+i}<\frac{1}{2003}$ for $i=1$ to 5, $\frac{7}{2009}<\frac{1}{2003}+\frac{1}{2004}+\ldots+\frac{1}{2009}<\frac{7}{2003}$ so $\frac{2003}{7}<\frac{1}{\frac{1}{2003}+\frac{1}{2004}+\ldots+\frac{1}{2009}}<\frac{2009}{7}$.

Or $\frac{2003}{7}=286+\frac{1}{7}$ and $\frac{2009}{7}=287$ so $286<\frac{1}{\frac{1}{2003}+\frac{1}{2004}+\ldots+\frac{1}{2009}}<287$.

Answer: 286.

22

It is easy to see that:

- $a_{ABK}=\frac{1}{2}a_{ABLJ}$
- $a_{ABD}+a_{DGJ}=\frac{1}{2}a_{ABLJ}$

So $a_{ABK}=a_{ABD}+a_{DGJ}$.

But:

- $a_{ABK}=a_{ABC}+a_{BCEF}+a_{EFHI}+a_{KHI}$
- $a_{ABD}+a_{DGJ}=a_{ABC}+a_{ACD}+a_{DEIJ}+a_{EFHI}+a_{FGH}$.

Hence:

$a_{ABC}+a_{BCEF}+a_{EFHI}+a_{KHI}=a_{ABC}+a_{ACD}+a_{DEIJ}+a_{EFHI}+a_{FGH}$

$a_{BCEF}+a_{KHI}=a_{ACD}+a_{DEIJ}+a_{FGH}$

$500+a_{KHI}=22+482+22$

$a_{KHI}=526-500=26$.

Answer: 26.

23 ◊

It isn't necessary to evaluate each number. Lets try to evaluate $\left(\sqrt[3]{77-20\sqrt{13}}+\sqrt[3]{77+20\sqrt{13}}\right)^3$.

$$\left(\sqrt[3]{77-20\sqrt{13}}\right)^3+\left(\sqrt[3]{77+20\sqrt{13}}\right)^3=77-20\sqrt{13}+77+20\sqrt{13}=154.$$

$$\sqrt[3]{77-20\sqrt{13}}\times\sqrt[3]{77+20\sqrt{13}}=\sqrt[3]{77^2-\left(20\sqrt{13}\right)^2}=\sqrt[3]{729}=9.$$

Or $\left(\sqrt[3]{77-20\sqrt{13}}+\sqrt[3]{77+20\sqrt{13}}\right)^3=\left(\sqrt[3]{77-20\sqrt{13}}\right)^3+$.

$$3\left(\sqrt[3]{77-20\sqrt{13}}\right)^2\left(\sqrt[3]{77+20\sqrt{13}}\right)+3\left(\sqrt[3]{77-20\sqrt{13}}\right)\left(\sqrt[3]{77+20\sqrt{13}}\right)^2+\left(\sqrt[3]{77+20\sqrt{13}}\right)^3$$

So $\left(\sqrt[3]{77-20\sqrt{13}}+\sqrt[3]{77+20\sqrt{13}}\right)^3=154+3\times9\times\left(\sqrt[3]{77-20\sqrt{13}}+\sqrt[3]{77+20\sqrt{13}}\right).$

We note $A=\left(\sqrt[3]{77-20\sqrt{13}}+\sqrt[3]{77+20\sqrt{13}}\right)$ and so A is solution of the equation $A^3=27A+154$, we write $A^3-27A-154=0$.

Lets try to factorize it to a form $(A-\alpha)(A^2+\beta A+\gamma)$, α, β, γ integers.
As $154=7\times22$, and $\alpha\times\gamma=154$, lets try $\alpha=7$ and $\gamma=22$.
It falls $\beta=7$ and $A^3-27A+154=(A-7)(A^2+7A+22)$.
It is clear that for $A\geq0$, $A^2+7A+22>0$ so we need $A=7$.

So $\left(\sqrt[3]{77-20\sqrt{13}}+\sqrt[3]{77+20\sqrt{13}}\right)=7$

Answer: 7.

24

Lets cut {1; …; 2009} in 3 sets: {1; …; 999}, {1000; …1999} and {2000; …; 2009}.

First case: a number from the set {1; …999} can be written $\overline{abc}$ with a, b and c digits and $a+b+c=11$.
• With $a=0$, $b+c=11$ so as $0\leq b\leq9$ and $0\leq c\leq9$ consequently $2\leq b\leq9$ and $2\leq c\leq9$.
We deduce 8 solutions: 029, 038, 047, 056, 065, 074, 083, 092.
• With $a=1$, $b+c=10$ and so $1\leq b\leq9$ and $1\leq c\leq9$: there are 9 possibilities.
• With $a=2$, $b+c=9$ and so $0\leq b\leq9$ and $0\leq c\leq9$: there are 10 possibilities.
• With $a=3$, $b+c=8$ and so $0\leq b\leq8$: there are 9 possibilities.
• With $a=4$, $b+c=7$, so there are 8 possibilities.
…
• With $a=9$, $b+c=2$ so there are 3 possibilities.
So there are $8+9+10+9+8+7+6+5+4+3=69$ numbers in the first set.

Second case: a number from the set {1000; …; 1999} can be written $\overline{1\,abc}$ with a, b and c digits such that $a + b + c = 10$.
• With $a = 0$, $b + c = 10$ so as $1 \leq b \leq 9$: there are 9 possibilities.
• With $a = 1$, $b + c = 9$ and so $0 \leq b \leq 9$: there are 10 possibilities.
• With $a = 3$, $b + c = 8$ and so $0 \leq b \leq 8$: there are 9 possibilities.
• With $a = 4$, $b + c = 7$, so there are 8 possibilities.
…
• With $a = 9$, $b + c = 1$ so there are 2 possibilities.
Then there are 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 = 63 numbers in the set 2.

Third case: a number from the set {2000; …; 2009}: only 2009 satisfies the condition.

Finally we have 69 + 63 + 1 = 133 integers satisfying the condition.

Answer: 133.

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25 ◊

Developing $x + (1 + x)^2 + \ldots + (1 + x)^n = (n - 1) + (1 + 2 + 3 + \ldots + n)\,x + \ldots + x^n$ so we can deduce that $a_1 = (1 + 2 + \ldots + n) = \dfrac{n(n+1)}{2}$ and $a_n = 1$.
In fact $(1 + x)^k = 1 + k\,x + \ldots$whatever$\ldots + x^k$.

Remark:
We know better than that using the binomial theorem and the Pascal's triangle.
$(1 + x)^0 = 1$
$(1 + x)^1 = 1 + x$
$(1 + x)^2 = 1 + 2\,x + x^2$
$(1 + x)^3 = 1 + 3\,x + 3\,x^2 + x^3$
$(1 + x)^4 = 1 + 4\,x + 6\,x^2 + 4\,x^3 + x^4$
$(1 + x)^5 = 1 + 5\,x + 10\,x^2 + 10\,x^3 + 5\,x^4 + x^5$
…
$(1 + x)^n = 1 + n\,x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + n\,x^{n-1} + x^n$.
We can observe that the coefficient of x^i in $(1 + x)^k$ is given by the sum of the coefficient of x^{i-1} and x^i in $(1 + x)^{k-1}$, unless for the first one x^0 and the last one x^k.
We usually use the Pascal's triangle:

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

…

We also have the binomial theorem which is written $(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i\, b^{n-i}$ where $\binom{n}{i} = \dfrac{n!}{i!\,(n-i)!}$ and where $i! = 1 \times 2 \times \ldots \times i$.

Then for $x = 1$, we obtain $1 + 2^2 + 2^3 + \ldots + 2^n = a_0 + a_1 + a_2 + \ldots + a_{n-1} + a_n$.

So as $a_n = 1$ and $a_1 = \dfrac{n(n+1)}{2}$, we deduce that $a_0 + a_2 + \ldots + a_{n-1} = 2^2 + 2^3 + \ldots + 2^n - \dfrac{n(n+1)}{2}$.

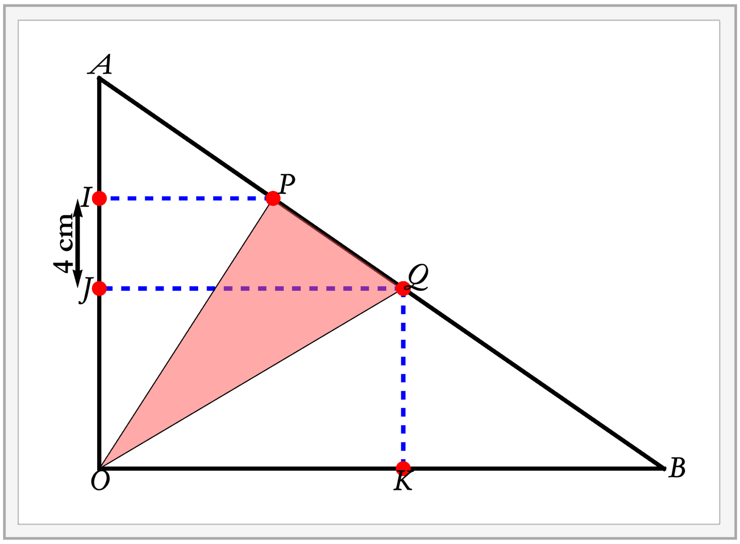
Thus as the condition is $a_0 + a_2 + \ldots + a_{n-1} = 60 - \dfrac{n(n+1)}{2}$, the integer n is such that $2^2 + 2^3 + \ldots + 2^n = 60$.

Finally, lets note that $2^2 + 2^3 + \ldots + 2^n = 2^2 (1 + 2 + 2^2 + \ldots + 2^{n-2})$ and as we suppose to know that $1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$, $2^2 + 2^3 + \ldots + 2^n = 4(2^{n-1} - 1)$ so $2^{n+1} - 4 = 60$.
Then $2^{n+1} = 64$ or $64 = 2^6$, and $n + 1 = 6$ and $n = 5$.

Answer: $n = 5$.

26

Lets consider the figure:



As $26\,AP = 22\,PQ = 11\,QB$, $AB = AP + \frac{26}{22} AP + \frac{26}{11} AP = \frac{50}{11} AP$ so $\frac{AP}{AB} = \frac{11}{50}$ and $AB = \frac{11}{26} QB + \frac{1}{2} QB + QB = \frac{25}{13} QB$ so $\frac{QB}{AB} = \frac{13}{25}$.

Using the Thales's theorem:

- $\frac{AP}{AB} = \frac{AI}{AO} = \frac{PI}{OB}$ and $\frac{AJ}{AB} = \frac{AQ}{AB}$ so $AI = \frac{11}{50} AO$, $PI = \frac{13 \times 11}{50}$ and $AJ = \frac{12}{25} AO$.

Or $IJ = AJ - AI = \left(\frac{12}{25} - \frac{11}{50}\right) AO = \frac{13}{50} AO$.

As $IJ = 4$, $AO = \frac{200}{13}$.

- $\frac{QK}{AO} = \frac{BQ}{BA}$ so $QK = \frac{13}{25} \times AO = \frac{13}{25} \times \frac{200}{13} = 8$.

Finally, $\mathcal{A}_{OPQ} = \mathcal{A}_{AOB} - \mathcal{A}_{OQB} - \mathcal{A}_{OPA} =$.

$$\frac{1}{2}(OA \times OB - OB \times QP - OA \times PI) = \frac{1}{2}\left(13 \times \frac{200}{13} - 8 \times 13 - \frac{13 \times 11}{50} \times \frac{200}{13}\right) = \frac{1}{2}(200 - 104 - 44) = 26$$

Answer: 26.

27 ◊

As x_1, x_2, x_3 and x_4 are the four roots of $x^4 + k x^2 + 90 x - 2009$, $(x - x_1)(x - x_2)(x - x_3)(x - x_4) = x^4 + k x^2 + 90 x - 2009$.

As $(x - x_1)(x - x_2)(x - x_3)(x - x_4) = x^4 - (x_1 + x_2 + x_3 + x_4) x^3 + (x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4) x^2 - (x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4) x + x_1 x_2 x_3 x_4$,

then $\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 = k \\ x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 = -90 \\ x_1 x_2 x_3 x_4 = -2009 \end{cases}$.

So:

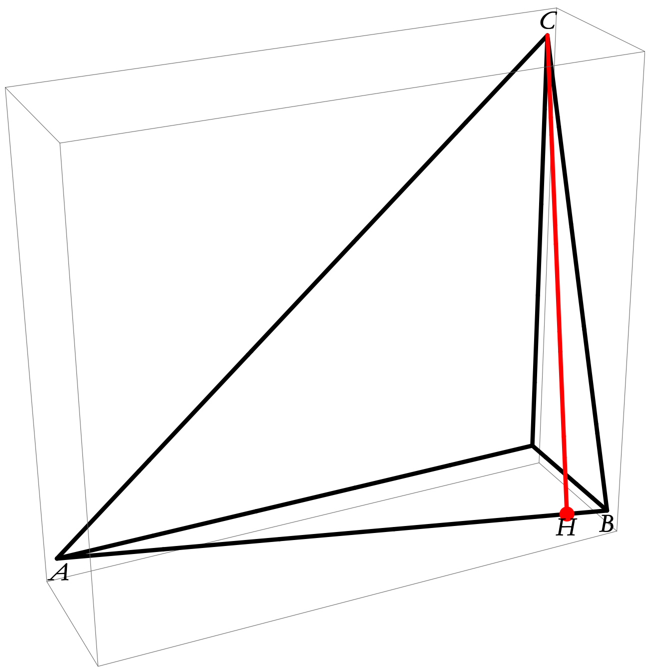
- as $x_1 x_2 = 49$ and $2009 = 49 \times 41$, $x_3 x_4 = -41$
- $x_3 + x_4 = -(x_1 + x_2)$
- as $x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 = x_1 x_2 (x_3 + x_4) + x_3 x_4 (x_1 + x_2) = 49 (x_3 + x_4) - 41 (x_1 + x_2)$, $x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 = -49 (x_1 + x_2) - 41 (x_1 + x_2) = -90 (x_1 + x_2)$ and as $x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4 = -90$, $x_1 + x_2 = 1$ and $x_3 + x_4 = -1$.

Finally $x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4 = 49 + x_1 (x_3 + x_4) + x_2 (x_3 + x_4) - 41 = 8 + (x_1 + x_2)(x_3 + x_4) = 8 - 1 \times (-1) = 7$.

Answer: $k = 7$.

28

We have the figure:



It is easy to get a_{OAB}, a_{OAC} and a_{OBC}.
Lets find a_{ABC}.

Using Pythagorea's Theorem, $AB = \sqrt{53}$, $AC = \sqrt{85}$ and $BC = \sqrt{40}$.
Lets consider H the foot of the altitude in ΔABC relative to the side $[AB]$.
Using again the Pythagorea's Theorem, $AC^2 = CH^2 + AH^2$ and $BC^2 = CH^2 + BH^2 = CH^2 + (AB - AH)^2 = CH^2 + AB^2 - 2\,AB\times AH + AH^2$.
Thus $AC^2 - BC^2 = -AB^2 + 2\,AB\times AH$ and then $AH = \dfrac{85 - 40 + 53}{2\times\sqrt{53}} = \dfrac{49}{\sqrt{53}}$.

Therefore $CH^2 = AC^2 - AH^2 = 85 - \dfrac{49^2}{53} = \dfrac{2104}{53}$ and $\mathcal{a}_{ABC} = \dfrac{1}{2}\times AB\times CH = \dfrac{1}{2}\times\sqrt{53}\times\sqrt{\dfrac{2104}{53}} = \dfrac{1}{2}\sqrt{2014}$

Hence $\mathcal{a}_{ABC}{}^2 + \mathcal{a}_{OAB}{}^2 + \mathcal{a}_{OAC}{}^2 + \mathcal{a}_{OBC}{}^2 =$.

$$\left(\frac{1}{2}\sqrt{2104}\right)^2 + \left(\frac{1}{2}\times7\times2\right)^2 + \left(\frac{1}{2}\times7\times6\right)^2 + \left(\frac{1}{2}\times6\times2\right)^2 = \frac{1}{4}(2104 + 196 + 1764 + 144) = \frac{1}{4}\times2\times2104 = 1052$$

Answer: 1052.

29

$\dfrac{n-10}{9\,n+11}$ is a non-zero reducible fraction.

Thus it exist an integer k, $k > 1$ and two integers a and b such that $\begin{cases} n - 10 = k\times a \\ 9\,n + 11 = k\times b \end{cases}$ and so $\begin{cases} n = a\,k + 10 \\ 9\,n = b\,k - 11 \end{cases}$.

We deduce $9\,a\,k + 90 = b\,k - 11$ and $k(b - 9\,a) = 101$.

As 101 is a prime integer, or $k = 1$ and $b - 9\,a = 101$ or $k = 101$ and $b - 9\,a = 1$.

Or $k > 1$ so $k = 101$ and $b - 9\,a = 1$.
We deduce that $n - 10 = 101\times a$ and so the least possible integer is for $a = 1$.
Hence $n = 101 + 10 = 111$.

Remark:

Whatever the value of a, positive integer, with then $b = 1 + 9\,a$, the fraction $\dfrac{n-10}{9\,n+11}$ is reducible.

Answer: $n = 111$.

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30 ◊

We have $x^2 + 2\,(m+5)\,x + (100\,m + 9) = (x + (m+5))^2 - \left[(m+5)^2 - 100\,m - 9\right]$.

Or $(m+5)^2 - 100\,m - 9 = m^2 - 90\,m + 16 = (m-45)^2 - 45^2 + 16 = (m-45)^2 - 2009$.

So $x^2 + 2\,(m+5)\,x + (100\,m+9) = (x + (m+5))^2 - \left[(m-45)^2 - 2009\right]$.

Assuming the equation have solutions, it means that $(m-45)^2 - 2009 \geq 0$, we have $x^2 + 2\,(m+5)\,x + (100\,m+9) = \left(x - \left[-(m+5) - \sqrt{(m-45)^2 - 2009}\right]\right)\left(x - \left[-(m+5) + \sqrt{(m-45)^2 - 2009}\right]\right)$.

Therefore, the equation have integer solutions if and only if $\sqrt{(m-45)^2 - 2009}$ is a perfect square.

It then must exist an integer n such as $n^2 = (m-45)^2 - 2009$, that gives $(m-45)^2 - n^2 = 2009$ and then $(m-45-n)(m-45+n) = 2009$.
As $2009 = 7^2 \times 41$, it falls:

- $m - 45 - n = 1$ and $m - 45 + n = 2009$: so $2(m-45) = 2010$ then $m = 1005 + 45 = 1050$.
  We also have $m - 45 - n = 2009$ and $m - 45 + n = 1$, which gives the same solution for m.
- $m - 45 - n = -1$ and $m - 45 + n = -2009$: so $2(m-45) = -2010$ then $m = -1005 + 45 = -960$.
- $m - 45 - n = 7$ and $m - 45 + n = 287$: so $m - 45 = 147$ and $m = 192$.
- $m - 45 - n = -1$ and $m - 45 + n = -287$: so $m - 45 = -147$ and $m = -102$.
- $m - 45 - n = 49$ and $m - 45 + n = 41$: so $m - 45 = 45$ and $m = 90$.
- $m - 45 - n = -49$ and $m - 45 + n = -41$: so $m - 45 = -45$ and $m = 0$.

Finally the smallest positive integer is $m = 90$.

Answer: $m = 90$.

31

Using the area formula, $AB \times CF = AC \times BE = BC \times AD$ so $AB \times CF = 12\,AC = 4\,BC$.

Then $BC = 3\,AC$.

Using the triangle inequality, $AB < AC + BC$ so $AB < 4\,AC$ and $BC < AB + AC$ so $2\,AC < AB$: $2\,AC < AB < 4\,AC$.

Finally as $CF = \dfrac{12\,AC}{AB}$, $\dfrac{12\,AC}{4\,AC} < CF < \dfrac{12\,AC}{2\,AC}$ so $3 < CF < 6$.
The largest integer possible value is thus 5.

Answer: 5.

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32 ◊

A four digit number with two distincts pairs of repeated digits is $abab$ or $abba$ or $aabb$ with $1 \le a \le 9$ and $0 \le b \le 9$ and $a \ne b$.

First case: type $abab$.
Lets note that $abab = 101 \times ab$.
There are 9×9 possibilities (9 choices for a and as $a \ne b$, 9 choices left for b) less the multiple of 7.
As $99 = 14 \times 7 + 1$: there are then 14 multiples of 7 between 1 and 99.
Note that we already didn't count the numbers 07 ($a \ne 0$) and 77 ($a \ne b$). So there are 12 multiples of 7 of the form ab with $1 \le a \le 9$ and $0 \le b \le 9$ and $a \ne b$.
Finally we count $81 - 12 = 69$ four digits integers of the form $abab$ divisible by 101 but not by 7.

Remark:
A four digits integer divisible by 101 is clearly of the form $abab$. So for the next cases, we only have to respect the second condition: divisible by 7.

Second case: type $abba$.
$abba = a00a + bb0 = 1001 \times a + 110 \times b = 91 \times 11\,a + 11 \times 10\,b = 11\,(13 \times 7 \times a + 10\,b)$.
As 11 is not divisible by 7 and as $91\,a$ is divisible by 7, $abba$ is divisible by 7 if and only if $10\,b$ is divisible by 7.
As 10 isn't divisible by 7, b must be divisible by 7: so $b = 0$ or $b = 7$.
It follows that there are $9 \times 2 - 1 = 17$ possibilities (9 possibilities for a and 2 for b, less the case where $a = 7$ and $b = 7$).

Third case: type $a\,a\,b\,b$.
$a\,a\,b\,b = a\,a \times 1100 + b\,b \times 11 = 11\,(100\,a + b)$.
As 11 isn't divisible by 7, we are looking for the number of the form $100\,a + b$ divisible by 7.
• With $a = 1$, $100\,a + b = 100 + b$ and $b \neq 1$: 100, 102, 103, 104, 05, 106, 107, 108, 109.
Only 105 is divisible by 7.
• With $a = 2$, $100\,a + b = 200 + b$: 200, 201, 203, 204, 205, 206, 207, 208, 209.
Only 203 is divisble by 7.
Going on this way we find that $(a; b)$ is (1; 5), (2; 3), (3; 1), (3; 8), (4; 6), (5; 4), (6; 2), (6; 9), (7; 0), (8; 5) and (9; 3).
There are 11 possibilities.

Finally we count 69 + 11 + 17 = 97 possibilities.

Answer: 97.

33

We want $m\,n$ divisible by $33 = 3 \times 11$.

If $n = 33$, all m such that $1 \leq m \leq 33$ is convenient: there are 33 possibilities.

If $m = 33$, all n such that $33 \leq n \leq 40$ is convenient. There are 8 possibilities, less 1 ((33; 33) that we already count in the precedent case), so 7 possibilities to count.

Then lets suppose that $m \neq 33$ and $n \neq 33$.
So as 33 divides $m\,n$ or 3 divides m and 11 divides n or 11 divides m and 3 divides n.
Lets note that there are only 3 numbers divided by 11 between 1 and 40: 11, 22, 33.

If 3 divides m and 11 divides n, *so* $m = 3\,a$ and $n = 11\,b$ and thus $1 \leq 3\,a \leq 11\,b \leq 40$.
We can assure that $b \neq 3$ as $n = 11\,b$ is not divisible by 33 and $a \neq 11$ as $m = 3\,a$ isn't divisible by 33 neither.
If $b = 1$, $a = 1;\ 2;\ 3$.
If $b = 2$, $a = 1;\ 2;\ 3;\ 4;\ 5;\ 6;\ 7$.
There are 10 possibilities.

If If 11 divides m and 3 divides n, *so* $m = 11\,a$ and $n = 3\,b$ and thus $1 \leq 11\,a \leq 3\,b \leq 40$.
We can assure that $b \neq 11$ as $n = 3\,b$ is not divisible by 33 and $a \neq 3$ as $m = 11\,a$ isn't divisible by 33 neither.
If $a = 1$, $b = 4;\ 5;\ 6;\ 7;\ 8;\ 9;\ 10;\ 12;\ 13$.
If $a = 2$, $a = 8;\ 9;\ 10;\ 12;\ 13$.
There are 14 possibilities.

Finally we count 33 + 7 + 10 + 14 = 64 pairs.

Answer: 64.

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34 ◊◊

Comments:
We will count the number of solutions by induction, building relations between the different types of "good" sequences.
Lets note that if a 13-digit sequence ends by 0 or 4, there are no other choices for the second last digit: 1 if 0 and 3 if 4.
Identically, if the last digit is a 1 or a 3, the second last is 0 or 2 if 1 and 2 or 4 if 3.
These remarks drive us to build a 13-digit "good" sequences using the n-digit "good" sequences with $1 \leq n \leq 12$.

Lets note:
- A_n the number of "good" sequence of length n that ends by 0 or 4.
- B_n the number of "good" sequence of length n that ends by 1 or 3.
- C_n the number of "good" sequence of length n that ends by 2.

The preceding remarks shows that for all n:

• $A_{n+1} = B_n$: in fact in order to get a "good" sequence of length $n + 1$ ending by 0 or 4, we needed to build a "good" sequence of length n ending by 1 or 3.

So each "good" sequence of length n ending by 1 or 3 can be converted in a "good" sequence of length $n + 1$ ending by 0 or 4.

• $B_{n+1} = A_n + 2\,C_n$: in fact in order to get a "good" sequence of length $n + 1$ ending by 0 or 4, we can add 1 or 3 to each "good" sequence of length n ending by 0 or 4 and for each "good" sequence of length n ending by 2, we can build 2 "good" sequences of length $n + 1$ ending by 1 or 3.

• $C_{n+1} = B_n$: in order to obtain a "good" sequence of length $n + 1$, we needed to have a "good" sequence of length n ending eiter by 1 or 3.

So $B_{n+1} = A_n + 2\,C_n = B_{n-1} + 2\,B_{n-1} = 3\,B_{n-1}$ for $n \geq 2$.
It easy to evaluate B_1 and B_2.
The number of "good" sequences of length 1 ending by 1 or 3 is 2: "1" or "3".
The number of "good" sequences of length 2 ending by 1 or 3 is 4: "01", "21", "43", or "23".

Then we have $B_1 = 2$ and $B_2 = 4$ and $B_{n+1} = 3\,B_{n-1}$.
So $B_3 = 3\,B_1$, $B_5 = 3\,B_3 = 9\,B_1$ and $B_{2p+1} = 2\times 3^p$ and identically $B_{2p} = 4\times 3^{p-1}$.

The number of "good" sequences of length 13 is $A_{13} + B_{13} + C_{13} = B_{12} + B_{13} + B_{12} = 2\,B_{12} + B_{13} = 2\times 4\times 3^{6-1} + 2\times 3^6$.
Thus we count $8\times 243 + 2\times 729 = 3402$.

Answer: 3402.

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35 ◊◊

As $m\,n = 1\,446\,921\,630$ is a 10-digit number and as m and n are reverse, m and n are 5-digit numbers.
In fact the numbers m and n have the same numbers of digits so if one is 6 digits (the first digit being different of 0, the smallest one is 100000) the other one also and then the product is at least 11 digits.
If they are only 4 digits (the greatest one is 9999) the product will be less than 9 digits.

Remark:
Supposing that one of m and n is written $0\,b\,c\,d\,e$ so the other one is $e\,d\,c\,b\,0$.
It is fine for the last digit of 1446921630 but then the product is at the maximum $09\,999\times 99\,990$ which has only 9 digtis.
So we can assure that 0 is neither the first nor the last digit of m or n.

Lets decompose 1446921630 ino prime factors.
We know that 1446921630 can be divided by 2, 5, 9 ($1 + 4 + 4 + 6 + 9 + 2 + 1 + 6 + 3 = 36$).
$1\,446\,921\,630 = 90\times 16\,076\,907$ and 16076907 is also divisible by 9.
$16\,076\,907 = 9\times 1\,786\,323$ and 1786323 is divisible by 3.
$1\,786\,323 = 3\times 595\,441$ not divisible by 2, 3 or 5.
We try with 7: it works! $595\,441 = 7\times 85\,063$.
85063 is not divisible by 7 so we try 11: $85\,063 = 11\times 7733$ and $7733 = 11\times 703$.
703 is not divisible by 11 nor 13 nor 17 but by 19. $703 = 19\times 37$ and 37 is prime.
Finally $1\,446\,921\,630 = 2\times 3^5\times 5\times 7\times 11^2\times 19\times 37$.
As $m\,n = 1\,446\,921\,630 = 2\times 3^5\times 5\times 7\times 11^2\times 19\times 37$, so we know that m and n are product of 2, 3, 5, 7, 11, 19 or 37 and that's all.

As the last digit of $m\,n$ is 0 we can deduce that either $5 \mid m$ and $2 \mid n$ or $2 \mid m$ and $5 \mid n$.
In fact as we remarked already, $2\times 5 = 10$ can not divide m or n.
As we did not choose m or n yet, lets suppose $5 \mid m$ and $2 \mid n$ and we know that 2 does not divide m and 5 does not divide n.
So the last digit of m is 5: $m = e\,d\,c\,b\,5$ and $n = 5\,b\,c\,d\,e$.
Next as the first digit of $m\,n = 1\,446\,921\,630$ is 1 and the last digit of n is 2.
(In fact the first digit of $m\,n = 1\,446\,921\,630$ is given by the product 5 (first digit of n) and 2, 4, 6 or 8 first digit of m).
Thus $m = 2\,d\,c\,b\,5$ and $n = 5\,b\,c\,d\,2$.

As $3^5 \mid m\,n$ so we know that $9 \mid m$ or $9 \mid n$.
In fact as the sums of the digits of m and n are equal, if one is divisible by 3 the other one also and so as we have "five factors 3", one is divisible by 9 so the other one also.

Trick:
It is easy to check the divisibility by 11: the difference between the sum of the odd-numbered digits and the even-numbered digits, counted from right to left is a multiple of 11.

In our case we deduce that $(2 + c + 5) - (b + d)$ is divisible by 11 as 11 is one of the prie factor of $m\,n$.
The difference of the sum is the same for both numbers m and n so 11 divide m and 11 divide n.

It follows that both m and n are divisible by $11 \times 9 = 99$.
Thus as n is even, $n = 198\,k$.
As the last digit of n is 2, the last digit of k is 4 ($4 \times 8 = 32$) or 9 ($8 \times 9 = 72$).
But as we know, k can not be even. Or n will be divisible by 4.
So the last digit of k is 9.
As $n = 198\,k$, k divide $m\,n = 1\,446\,921\,630 = 2 \times 3^5 \times 5 \times 7 \times 11^2 \times 19 \times 37$ but the factors 2, 5, 4 factors 3, and 11's are already taken.
So the remaining factors for k are 3, 7, 19 and 37.
Lets try it out (the last digit of k is 9).

- $3 \times 7 = 21$: not convenient
- $3 \times 37 = \ldots 1$: not convenient
- $7 \times 37 = \ldots 9$: convenient
- …

We obtain that the possible values for k are $7 \times 37 = 259$, $3 \times 7 \times 19 = 399$, $3 \times 19 \times 37 = 2109$.

But we also know that $n = 5\,b\,c\,d\,2$ so $50\,000 < 198\,k < 60\,000$ and then $253 \leq k \leq 303$.
Only $k = 7 \times 37$ is suitable.

Finally $n = 198 \times 259 = 51\,282$ and $m = 28\,215$.

Thus $m + n = 51\,282 + 28\,215 = 79\,497$.

Answer: 79497.