Maths Lab: SMO Senior 2009.

Maths Lab: Elements of solutions.

The ◊ indicates difficult questions.

1

Answer (C).

Lets note 7 + 26 = 33.

A line d is at the distance 7 from A, if and only if line d is tangent to the circle of center A with radius 7.
Equally, a line d is at the distance 23 from B , if and only if line d is tangent to the circle of center B with radius 23.

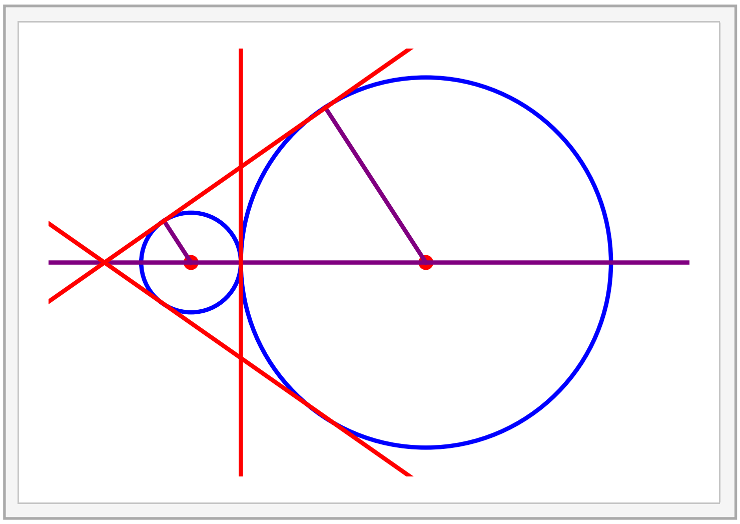
So the problem is to enumerate the number of lines which are tangent to the both circles.
As 7 + 26 = 33, the two circles are tangent. So we have already one line.

Another line corresponding to the situation, must be perpendicular to the radius of the two circles.
It implies that these two radius must be parallel and the tangent line is such that we have two right-angled triangles, in a Thales's configuration.

So we deduce there are 2 more lines satisfying the conditions.

On the figure below, the three tangents in red.

Out[28]=



2

Answer (A).

$y = (17 - x)(19 - x)(17 + x)(19 - x)$
$y = (17^2 - x^2)(19^2 - x^2)$
$y = 17^2\, 19^2 - (17^2 + 19^2)\, x^2 + x^4$
$y = \left(x^2 - \frac{17^2+19^2}{2}\right)^2 - \frac{(17^2+19^2)^2}{4} + 17^2\, 19^2.$

So the minimum value is $-\frac{\left(17^2+19^2\right)^2}{4} + 17^2\,19^2$.

$$-\frac{\left(17^2+19^2\right)^2}{4} + 17^2\,19^2 = -\frac{17^4}{4} - \frac{19^2}{4} - 2\,\frac{17^2\,19^2}{4} + 17^2\,19^2$$

$$-\frac{\left(17^2+19^2\right)^2}{4} + 17^2\,19^2 = -\left(\frac{17^4}{4} + \frac{19^2}{4} - 2\,\frac{17^2\,19^2}{4}\right)$$

$$-\frac{\left(17^2+19^2\right)^2}{4} + 17^2\,19^2 = \frac{-1}{4}\left(17^2-19^2\right)^2$$

$$-\frac{\left(17^2+19^2\right)^2}{4} + 17^2\,19^2 = -\frac{1}{4}\left((17-19)\,(17+19)\right)^2$$

$$-\frac{17^4}{4} - \frac{19^2}{4} - \frac{17^2\,19^2}{2} = -\frac{1}{4}\,(2\times 36)^2 = -\frac{5184}{4} = -1296.$$

3

Answer (A).

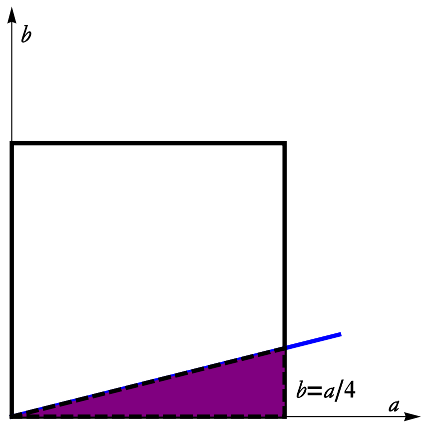
$x^2 - \sqrt{a}\;x + b = 0$ can be written $\left(x - \frac{\sqrt{a}}{2}\right)^2 - \frac{a}{4} + b = 0$.

In order to have real roots to this equation, it is necessary that $-\frac{a}{4} + b \le 0$.

In fact then $-\frac{a}{4} + b$ can be written $-\left(\sqrt{\frac{a}{4} - b}\right)^2$ and the equation become $\left(x - \frac{\sqrt{a}}{2} - \sqrt{\frac{a}{4} - b}\right)\left(x - \frac{\sqrt{a}}{2} + \sqrt{\frac{a}{4} - b}\right) = 0$.

It follows that $a \ge 4\,b$.

Then the probability is given by all the possible ordered pairs $(a;\,b)$ such that $a \ge 4\,b$ divided by all the possible ordered pairs $(a;\,b)$.
As the numbers a and b are reals, lets represent the situation by a figure.
The square of side 1 represents all the possible ordered pairs and the shaded area all the convenient pairs.



The probability is the ratio of the shaded area with the area of the square.

The probability is then $\frac{1}{8}$.

4

Answer (D).

Assuming $y \geq 0$, we deduce $x = 7 \geq 0$ and then $7 + 7 + 5\,y = 2$ so $y = \dfrac{-12}{5} < 0$: absurd.

So $y < 0$ thus $x - 2\,y = 7$.

Then assuming that $x \leq 0$, we deduce $5\,y = 2$ so $y = \dfrac{2}{5} > 0$: absurd.

Thus $x > 0$.

Finally x and y are solutions of the system $\begin{cases} x - 2\,y = 7 \\ 2\,x + 5\,y = 2 \end{cases}$ which gives $\begin{cases} y = \dfrac{-4}{3} \\ x = \dfrac{13}{3} \end{cases}$.

Therefore $x + y = \dfrac{13}{3} - \dfrac{4}{3} = 3$.

5

Answer: (C).

As $\sin(A) = \dfrac{3}{5}$, then $\cos(A) = \sqrt{1 - \sin^2(A)} = \sqrt{1 - \left(\dfrac{3}{5}\right)^2} = \pm\dfrac{4}{5}$.

Identically, as $\cos(B) = \dfrac{5}{13}$, then $\sin(B) = \sqrt{1 - \left(\dfrac{5}{13}\right)^2} = \pm\dfrac{12}{13}$.

But as $0 < B < 180\,^{\circ}$, then $\sin(B) > 0$ so $\sin(B) = \dfrac{12}{13}$.

If $\cos(A) = -\dfrac{4}{5}$, as $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B) = \dfrac{3}{5}\times\dfrac{5}{13} - \dfrac{4}{5}\times\dfrac{12}{13} < 0$: that is not possible as $0 < A + B < 180\,^{\circ}$.

Thus $\cos(A) = \dfrac{4}{5}$.

Then, as $C = 180\,^{\circ} - (A + B)$, $\cos(C) = \cos(180\,^{\circ} - (A + B)) = -\cos(A + B)$, so $\cos(C) = -(\cos(A)\cos(B) - \sin(A)\sin(B))$ and finally $\cos(C) = -\left(\dfrac{4}{5}\times\dfrac{5}{13} - \dfrac{3}{5}\times\dfrac{12}{13}\right) = \dfrac{16}{65}$.

6

Answer (E).

$\mathcal{a}_{ABE} = \mathcal{a}_{ADE} + \mathcal{a}_{DBE}$ and $\mathcal{a}_{BDFE} = \mathcal{a}_{DEF} + \mathcal{a}_{DBE}$.

So as $\mathcal{a}_{ABE} = \mathcal{a}_{BDFE}$, $\mathcal{a}_{ADE} = \mathcal{a}_{DEF}$.

Lets note that $\mathcal{A}_{ADE} = \frac{1}{2} DE \times h$ where h is the altitude of ADE issued by A so the distance from A to line (DE) and that $\mathcal{A}_{DEF} = \frac{1}{2} DE \times h'$ where h' is the altitude of DEF issued by F so the distance from F to line (DE).

From $\mathcal{A}_{ADE} = \mathcal{A}_{DEF}$, we get $h = h'$.

It implies that two points of the line (AF) are at an equal distance from line (DE).

In conclusion $(AF) \,//\, (DE)$.

According to the Thales's theorem, $\frac{CE}{CB} = \frac{AD}{AB} = \frac{3}{8}$.

Then as the triangles AEC and ABC share the same altitude δ issued by A (perpendicular to line (BC)), $\frac{\mathcal{A}_{AEC}}{\mathcal{A}_{ABC}} = \frac{\frac{1}{2} CE \times \delta}{\frac{1}{2} BC \times \delta} = \frac{CE}{BC} = \frac{3}{8}$.

Hence $\mathcal{A}_{AEC} = \frac{3}{8} \mathcal{A}_{ABC} = 15$.

7

Answer (C).

Consider an integer i.

$$\frac{i+2}{i!+(i+1)!+(i+2)!} = \frac{i+2}{i!\,(1+(i+1)+(i+1)(i+2))} = \frac{i+2}{i!\,(i+2)(1+i+1)} = \frac{i+2}{i!\,(i+2)^2} = \frac{1}{i!\,(i+2)}.$$

Then

$$\frac{1}{i!\,(i+2)} = \frac{i+1}{i!\,(i+1)(i+2)} = \frac{i+1}{(i+2)!} = \frac{i+2-1}{(i+2)!} = \frac{1}{(i+1)!} - \frac{1}{(i+2)!}.$$

Therefore:

$$\frac{3}{1!+2!+3!} + \frac{4}{2!+3!+4!} \ldots + \frac{22}{20!+21!+22!} = \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \ldots + \left(\frac{1}{21!} - \frac{1}{22!}\right)$$

And finally

$$\frac{3}{1!+2!+3!} + \frac{4}{2!+3!+4!} \ldots + \frac{22}{20!+21!+22!} = \frac{1}{2!} - \frac{1}{22!} = \frac{1}{2} - \frac{1}{22!}.$$

Remark:
It is quite clear that one must try to find such a trick (a difference between consecutive numbers) in order to simplify the sum.

8

Answer (D).

There are $\frac{8\times7\times6\times5}{1\times2\times3\times4} = 70$ ways of combining 4 numbers of the list $\{1; \ldots; 8\}$.

($8\times7\times6\times5$ ways counting with order and each case is repeated $1\times2\times3\times4$).

Lets note that $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$.
For a combination, there are then 3 possibilities:

- or the sum is equal to 18 and so the sum of the remaining 4 is also equal to 18;
- or the sum is more than 18 and so the sum of the remaining for is less than 18;
- or the sum is less than 18 and so the sum of the remaining 4 is more than 18.

We can deduce that to each sum more than 18 there are a correspond ant combination for which the sum is less than 18.
Lets note n the number of combinations for which the sum is 18, the number of combinations for which the sum is more than 18 is then $\frac{70-n}{2}$.

Lets count the number of combination for which the sum is equal to 18.
We note such a combination $\{r_1; r_2; r_3; r_4\}$, assuming $r_1 < r_2 < r_3 < r_4$.
With $r_4 = 8$:

- with $r_3 = 7$, $r_1 = 1$ and $r_2 = 2$;
- with $r_3 = 6$, $r_1 = 1$ and $r_2 = 3$;
- with $r_5 = 5$, $r_1 = 1$ and $r_2 = 4$ or $r_1 = 2$ and $r_2 = 3$.

Then we can deduce that there are 8 combinations for which the sum is 18.
In fact, if $r_4 < 8$, then the correspondent combination contains the number 8 and is one of the above.

Finally, there are $\frac{70-8}{2} = 31$ combinations.

Remark:
One can also enumerate the possibilities.

9

Answer (B).

The sum of the angles of a triangle is 180º.
Lets note x the smallest one. Then the largest one is $3x$.
The remaining angle α then verifies $x \leq \alpha \leq 3x$.
So the sum of the three angles verifies $5x \leq 4x + \alpha \leq 7x$.
Then $5x \leq 180 \leq 7x$.
So $x \leq 36$ and $x > 25$.
There are 11 possibilities.

But we know that $3x < 90°$ so $x < 30°$.
4 possibilities remaining.

10

Answer (B).

As $AD = DC$, and D is on the circle of diameter $[AC]$, the triangle ACD is right-angled and isosceles in D.
So $\measuredangle DAC = 45°$.
Using the theorem of the inscribed angle in a circle, $\measuredangle DBC = \measuredangle DAC = 45°$.
Thus the triangle BED is rigth-angled in E with another angle measuring 45º: he also is isosceles in E.
Then $BE = DE$.

Next as as 2 lines perpendicular to the same line (BE), we can deuce that the altitude issued by D in the triangle BCD has length $BE = DE$ ($BEDH$, where H is the foot of the altitude issued by D, is a square).

Lets note r the radius of the circle and $x = DE$.
Lets now express the area of quadrilateral $ABCD$.

• $\mathcal{A}_{ABCD} = \mathcal{A}_{ADE} + \mathcal{A}_{BED} + \mathcal{A}_{BCD} = \frac{1}{2} AE \times DE + \frac{1}{2} BE \times DE + \frac{1}{2} AC \times DE$.

So $\mathcal{A}_{ABCD} = \frac{1}{2}(AB - x)\,x + \frac{1}{2}x^2 + \frac{1}{2}AC\,x = \frac{1}{2}x(AB + BC)$.

We deduce $x = \dfrac{48}{AB + BC}$.

• $\mathcal{A}_{ABCD} = \mathcal{A}_{ABC} + \mathcal{A}_{ACD} = \frac{1}{2} AB \times AC + \frac{1}{2} AC \times DO$ (O center of the circle).

As triangle ACD is right-angled and isosceles in D, $DO = r$ and then $\mathcal{A}_{ACD} = r^2$ and so $\mathcal{A}_{ABCD} = \frac{1}{2} AB \times AC + r^2$.

We deduce $AB \times AC = 2\,(24 - r^2)$.

Using Pythagoras's Theorem, $(2\,r)^2 = AB^2 + AC^2$.
Then as $(AB + BC)^2 = AB^2 + 2\,AB \times BC + BC^2$ gives $(AB + BC)^2 = 4\,r^2 + 4\,(24 - r^2) = 96$.
So $AB + BC = \sqrt{96} = 4\sqrt{6}$.

Finally $x = \dfrac{48}{4\sqrt{6}} = 2\sqrt{6}$.

11

$\left(2008^3 + 3 \times 2008 \times 2009 + 1\right)^2 = \left(2008^3 + 3 \times 2008^2 + 3 \times 2008 + 1\right)^2 = \left((2008 + 1)^3\right)^2 = 2009^6$.

As $2009 = 7^2 \times 41$ so $\left(2008^3 + 3 \times 2008 \times 2009 + 1\right)^2 = 7^{12} \times 41^6$.

Thus there are $13 \times 7 = 91$ divisors of $\left(2008^3 + 3 \times 2008 \times 2009 + 1\right)^2$.

In fact each 7^i, $i = 0; \ldots; 12$ can be multiply by 13^j, $j = 0; \ldots; 6$ in order to obtain a divisor

Answer: 91.

12

Note:

For all positive number a, $a \neq 1$ and all real x, $x > 0$, $\mathrm{Log}_a(x) = \dfrac{\ln(x)}{\ln(a)} = \dfrac{\mathrm{Log}(x)}{\mathrm{Log}(a)}$ where ln = Log is the natural logarithm.

The natural logarithm is the only function such as $f(a\,b) = f(a) + f(b)$ for all positive reals and $\ln(e) = 1$ where e is the irrational number such that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \ldots$.

Then for all positive real a, $a \neq 1$, and all reals x, y positive:

- $\mathrm{Log}_a(x\,y) = \mathrm{Log}_a(x) + \mathrm{Log}_a(y)$ $\quad \ln(x\,y) = \ln(x) + \ln(y)$
- $\mathrm{Log}_a\left(\frac{1}{x}\right) = -\mathrm{Log}_a(x) \ln\left(\frac{1}{x}\right) = -\ln(x)$
- $\mathrm{Log}_a\left(\frac{x}{y}\right) = \mathrm{Log}_a(x) - \mathrm{Log}_a(y)$ $\quad \ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$
- For all real n non zero, $\mathrm{Log}_a(x^n) = n\,\mathrm{Log}_a(x)$ $\quad \ln(x^n) = n\ln(x)$.

Lets note:

$\text{Log}_{a^2 b}\left(\frac{c}{a}\right) = \frac{\ln(c) - \ln(a)}{2\ln(a) + \ln(b)}$ so $1 + \text{Log}_{a^2 b}\left(\frac{c}{a}\right) = 1 + \frac{\ln(c) - \ln(a)}{2\ln(a) + \ln(b)} = \frac{\ln(a) + \ln(b) + \ln(c)}{2\ln(a) + \ln(b)}$ and $\frac{1}{1 + \text{Log}_{a^2 b}\left(\frac{c}{a}\right)} = \frac{2\ln(a) + \ln(b)}{\ln(a) + \ln(b) + \ln(c)}$.

Similarly $\frac{1}{1 + \text{Log}_{b^2 c}\left(\frac{a}{b}\right)} = \frac{2\ln(b) + \ln(c)}{\ln(a) + \ln(b) + \ln(c)}$ and $\frac{1}{1 + \text{Log}_{c^2 a}\left(\frac{b}{c}\right)} = \frac{2\ln(c) + \ln(a)}{\ln(a) + \ln(b) + \ln(c)}$.

Finally $\frac{1}{1 + \text{Log}_{a^2 b}\left(\frac{c}{a}\right)} + \frac{1}{1 + \text{Log}_{b^2 c}\left(\frac{a}{b}\right)} + \frac{1}{1 + \text{Log}_{c^2 a}\left(\frac{b}{c}\right)} = \frac{2\ln(a) + \ln(b)}{\ln(a) + \ln(b) + \ln(c)} + \frac{2\ln(b) + \ln(c)}{\ln(a) + \ln(b) + \ln(c)} + \frac{2\ln(c) + \ln(a)}{\ln(a) + \ln(b) + \ln(c)}$

and $\frac{1}{1 + \text{Log}_{a^2 b}\left(\frac{c}{a}\right)} + \frac{1}{1 + \text{Log}_{b^2 c}\left(\frac{a}{b}\right)} + \frac{1}{1 + \text{Log}_{c^2 a}\left(\frac{b}{c}\right)} = 3$.

Answer: 3.

13

For all integer positive integer n, $n!\,n = n!\,(n+1-1) = n!\,(n+1) - n = (n+1)! - n!$.
Thus $1!\times 1 + 2!\times 2 + \ldots + 286!\times 286 = (2! - 1!) + (3! - 2!) + \ldots + (287! - 286!) = 287! - 1!$

Note that $2009 = 7\times 287$ so as $287 \neq 1\times 2\times \ldots\times 7\times \ldots\times 287$ then $287!$ is divisible by 2009.
It exists an integer k such that $287! = 2009\,k$ so $287! - 1! = 2009\,(k-1) + 2009 - 1 = 2009\,(k-1) + 2008$.
The remainder is 2008.

Answer: 2008.

14

Reminder:
$x^{1/3} = \sqrt[3]{x}$ and for all integer n, $n \neq 0$ and all real x positive (0 included), $x^{1/n} = \sqrt[n]{x}$.
In fact if n is even, $x \geq 0$ and if n is odd, x real is enough.

Lets note $S = \left(25 + 10\sqrt{5}\right)^{1/3} + \left(25 - 10\sqrt{5}\right)^{1/3} = \sqrt[3]{25 + 10\sqrt{5}} + \sqrt[3]{25 - 10\sqrt{5}}$.

T h e n
$S^3 =$

$$\left(\sqrt[3]{25 + 10\sqrt{5}}\right)^3 + 3\left(\sqrt[3]{25 + 10\sqrt{5}}\right)^2\left(\sqrt[3]{25 - 10\sqrt{5}}\right) + 3\left(\sqrt[3]{25 + 10\sqrt{5}}\right)\left(\sqrt[3]{25 - 10\sqrt{5}}\right)^2 + \left(\sqrt[3]{25 - 10\sqrt{5}}\right)^3$$

so

$$S^3 = \left(\sqrt[3]{25 + 10\sqrt{5}}\right)^3 + 3\left(\sqrt[3]{25 + 10\sqrt{5}}\right)\left(\sqrt[3]{25 - 10\sqrt{5}}\right)\left(\sqrt[3]{25 + 10\sqrt{5}} + \sqrt[3]{25 - 10\sqrt{5}}\right) + \left(\sqrt[3]{25 - 10\sqrt{5}}\right)^3.$$

Yet, $\left(\sqrt[3]{25+10\sqrt{5}}\right)^3 + \left(\sqrt[3]{25-10\sqrt{5}}\right)^3 = 25+10\sqrt{5}+25-10\sqrt{5} = 50$ and

$\left(\sqrt[3]{25+10\sqrt{5}}\right)\left(\sqrt[3]{25-10\sqrt{5}}\right) = \sqrt[3]{\left(25+10\sqrt{5}\right)\left(25-10\sqrt{5}\right)} = \sqrt[3]{25^2-100\times 5} = \sqrt[3]{125} = 5.$

Finally $S^3 = 15S + 50$ so S is solution of the equation $S^3 - 15S - 50 = 0$.
Yet $S^3 - 15S - 50 = (S-5)(S^2+5S+10)$.
Thus or $S = 5$ or $S^2 + 5S + 10 = 0$.

As $S^2 + 5S + 10 = \left(S - \frac{5}{2}\right)^2 + \frac{15}{4} \geq \frac{15}{4} > 0$, $S^2 + 5S + 10 \neq 0$.

The only solution is 5.

Answer: 5.

15

Lets do it!
$a^3 - 503a - 500 = a(a^2 - 503) - 500$.

Yet $a^2 - 503 = \frac{1}{4}\left(1+7\sqrt{41}\right)^2 - 503 = \frac{1}{4}\left(1+2\times 7\sqrt{41}+2009-2012\right) = \frac{1}{2}\left(7\sqrt{41}-1\right) = \frac{7\sqrt{41}-1}{2}$.

Thus $a\left(a^2-503\right) = \left(\frac{1+7\sqrt{41}}{2}\right)\left(\frac{7\sqrt{41}-1}{2}\right) = \frac{1}{4}(2009-1) = \frac{2008}{4} = 502$.

So $a^3 - 503a - 500 = 502 - 500 = 2$.

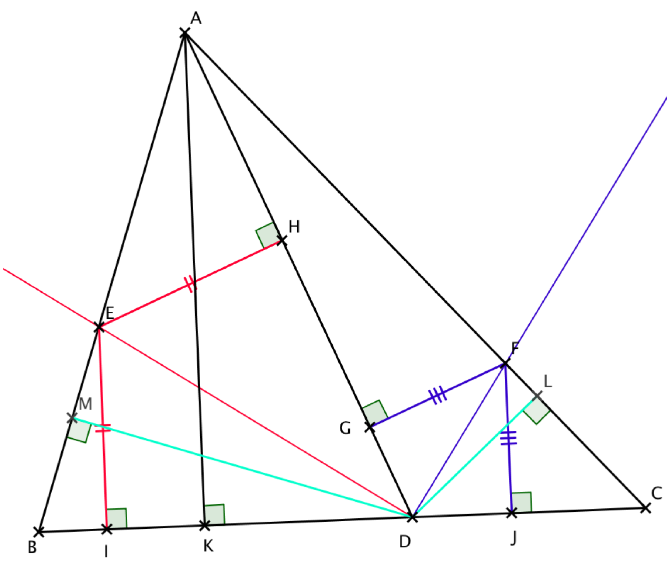
Finally $(a^3 - 503a - 500)^{10} = 2^{10} = 1024$.

Answer: 1024.

16

Reminder:
The angle bissector is the set of the points of the plan equidistant from each side of the angle.

We have the figure:



Lets note that the triangle *ABD* and *ADC* share the same altitude $[AK]$.

Then $\dfrac{\mathcal{a}_{ABD}}{\mathcal{a}_{ADC}} = \dfrac{\frac{1}{2} BD \times AK}{\frac{1}{2} DC \times AK} = \dfrac{BD}{DC}$.

As reminded, $EH = EI$ and $GF = FJ$ so:

- $\dfrac{\mathcal{a}_{ADE}}{\mathcal{a}_{BDE}} = \dfrac{\frac{1}{2} AD \times EH}{\frac{1}{2} BD \times EI} = \dfrac{AD}{BD}$.
- $\dfrac{\mathcal{a}_{ADF}}{\mathcal{a}_{CDF}} = \dfrac{\frac{1}{2} AD \times EH}{\frac{1}{2} CD \times EI} = \dfrac{AD}{CD}$.

Moreover the triangles *BDE* and *ADE* share the same altitude $[DM]$ as well as the triangles C*DF* and *ADF* share the same altitude $[DL]$.
So:

- $\dfrac{\mathcal{a}_{ADE}}{\mathcal{a}_{BDE}} = \dfrac{\frac{1}{2} AE \times DM}{\frac{1}{2} BE \times DM} = \dfrac{AE}{BE}$, we deduce $\dfrac{AE}{BE} = \dfrac{AD}{BD}$.
- $\dfrac{\mathcal{a}_{ADF}}{\mathcal{a}_{CDF}} = \dfrac{\frac{1}{2} AF \times DL}{\frac{1}{2} CF \times DL} = \dfrac{AF}{CF}$, we deduce $\dfrac{AF}{CF} = \dfrac{AD}{CD}$.

Finally $\dfrac{AE}{EB} \times \dfrac{BD}{DC} \times \dfrac{CF}{FA} = \dfrac{AD}{BD} \times \dfrac{BD}{DC} \times \dfrac{CD}{AD} = 1$.

Answer: 1.

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17 ◊

Reminder:

- We note "tan" for the function tangent defined by $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, $x \neq \frac{\pi}{2}$ (modulo 180 °).
- We call cotangent and note "cot" the function defined by $\cot(x) = \dfrac{\cos(x)}{\sin(x)} = \dfrac{1}{\tan(x)}$ for $x \neq 0$ (modulo 180 °).

It is not very common in Europe, at least in secondary education.

- trigonometric function have a lot of properties, particularly additional properties:
  - $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$
  - $\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$
  - $\tan(a + b) = \dfrac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$

Lets note that for $a + b = 45\,°$, $\tan(a + b) = 1$ so $\dfrac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)} = 1$ and then $1 - \tan(a) - \tan(b) = \tan(a)\tan(b)$.

Therefore as $\cot(a) - 1 = \dfrac{1}{\tan(a)} - 1 = \dfrac{1 - \tan(a)}{\tan(a)}$, if moreover $a + b = 45\,°$,
$(\cot(a) - 1)(\cot(b) - 1) = \left(\dfrac{1 - \tan(a)}{\tan(a)}\right)\left(\dfrac{1 - \tan(b)}{\tan(b)}\right) = \dfrac{1 - \tan(a) - \tan(b) + \tan(a)\tan(b)}{\tan(a)\tan(b)} = \dfrac{2\tan(a)\tan(b)}{\tan(a)\tan(b)} = 2$.

Thus as $25\,° + 20\,° = 45\,°$, $24\,° + 21\,° = 45\,°$ and $23\,° + 22\,° = 45\,°$,
$(\cot(25\,°) - 1)(\cot(24\,°) - 1)(\cot(23\,°) - 1)(\cot(22\,°) - 1)(\cot(21\,°) - 1)(\cot(20\,°) - 1) = 2 \times 2 \times 2 = 8$.

Answer 8.

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18 ◊

Remarking that $(a+1)(b+1) = a\,b + a + b + 1$ then looking for the 2-element subsets $\{a; b\}$ with $1 \le a \le 100$ and $1 \le b \le 100$ such as $a\,b + a + b$ is divisible by 7, is equivalent to look for the 2-element subsets $\{x; y\}$ with $2 \le x \le 101$ and $2 \le y \le 101$such that $x\,y = 7\,k + 1$, $k \in \mathbb{N}$.

We could try to enumerate the different possibilities, but it quickly appears an exhausting task.
Lets observe the properties of divisibility.
It is clear that neither x nor y can be a multiple of 7.
Then there are 6 more classes of numbers regarding the divisibility by 7: the $7\,p + 1$'s, the $7\,p + 2$'s, …, the $7\,p + 6$'s.
Lets determine which combination of product gives a $7\,k + 1$.
Writing $x = 7\,p + i$ and $y = 7\,p' + j$ with $1 \le i \le 6$ and $1 \le j \le 6$ then
$x\,y = (7\,p + i)(7\,p' + j) = 49\,p\,p' + 7\,p\,j + 7\,p'\,i + i\,j = 7\,k + i\,j$.
Consequently, the remainder of $x\,y$ is the remainder of $i\,j$.

- with $i = 1$, only if $j = 1$, the remainder dividing by 7 is 1.
- with $i = 2$, it falls $j = 4$,
- with $i = 3$, it falls $j = 5$,
- with $i = 4$, it falls $j = 2$,
- with $i = 5$, it falls $j = 3$,
- with $i = 6$, it falls $j = 6$.

There are:

- fourteen $7\,p + 1$'s in the set $\{2; …; 101\}$,
- fifteen $7\,p + 2$'s in the set $\{2; …; 101\}$,
- fifteen $7\,p + 3$'s in the set $\{2; …; 101\}$,
- fourteen $7\,p + 4$'s in the set $\{2; …; 101\}$,
- fourteen $7\,p + 5$'s in the set $\{2; …; 101\}$,
- fourteen $7\,p + 6$'s in the set $\{2; …; 101\}$.

Noticing that the subset $\{a, b\} = \{b, a\}$ and that $a \ne b$, it follows that the cases $i = 2$ and $j = 4$ and $i = 4$ and $j = 2$ are one case as well as $i = 3$ and $j = 5$ and $i = 5$ and $j = 3$.

- there are $\dfrac{14 \times 13}{2} = 91$ possibilities with $i = j = 1$ such that $x \ne y$ as well as there are $\dfrac{14 \times 13}{2} = 91$ possibilities with $i = j = 6$ and $x \ne y$.
- there are $15 \times 14 = 210$ possibilities with $i = 2$ and $j = 4$ (one can notice that in this case there can't be any $x = y$ as the $7\,p + 2$'s and $7\,p + 4$'s are distinct subsets of $\{2; …; 101\}$
- there are $15 \times 14 = 210$ possibilities with $i = 5$ and $j = 3$.

Finally there are $91 + 91 + 210 + 210 = 602$ possibilities.

Answer: 602.

19

A second degree polynomial equation has two roots (eventually equal) and then calling x' the second root, $X^2 - 15\,X + 1 = (X - x)(X - x')$ so we deduce that $x\,x' = 1$, namely $x' = \dfrac{1}{x}$ and that $-(x + x') = -15$ so finally $x + \dfrac{1}{x} = 15$.

Then $\left(x+\frac{1}{x}\right)^2 = x^2+2+\frac{1}{x^2} = 15^2$ so $x^2+\frac{1}{x^2} = 223$ and finally $\left(x^2+\frac{1}{x^2}\right)^2 = x^4+2+\frac{1}{x^4} = 223^2$.

Thus $x^4+\frac{1}{x^4} = 49\,727$.

Answer: 49727.

20

As (BE) bisects $\measuredangle CBA$ then E is equidistant from (BC) and (AB).

Thus $\frac{\mathcal{A}_{ABE}}{\mathcal{A}_{BCE}} = \frac{\frac{1}{2}AB\times d(E;(AB))}{\frac{1}{2}BC\times d(E;(BC))} = \frac{AB}{BC} = \frac{1}{4}$.

Yet we also notice that the triangles have the same altitude h through vertex B regarding the base $[AE]$ and $[CE]$.

So $\frac{\mathcal{A}_{ABE}}{\mathcal{A}_{BCE}} = \frac{\frac{1}{2}AE\times h}{\frac{1}{2}CE\times h} = \frac{AE}{CE}$.

Consequently, $\frac{AE}{CE} = \frac{1}{4}$.

As (BE) // (DF) and (AB) // (EF), using the alternate interior angles properties, (DF) bisects $\measuredangle CFE$, so $\frac{DE}{CD} = \frac{EF}{CF}$.

As (AB) // (EF), using Thales's Theorem, $\frac{EF}{AB} = \frac{CF}{CB}$ so $\frac{EF}{CF} = \frac{AB}{CB} = \frac{1}{4}$.

Finally $\frac{DE}{CD} = \frac{AE}{CE} = \frac{1}{4}$so $4\,DE = CD$ and $4\,AE = CE$.

As $CE = CD + DE = 5\,DE$ then $4\,AE = 5\,DE$ and $AD = AE + DE = 13,5$.

The lengths AE and DE are solutions of the system $\begin{cases} AE + DE = 13,5 \\ 4\,AE = 5\,DE \end{cases}$.

Thus $AE = 7,5$ and $DE = 6$.

At least, $CD = 24$.

Answer: 24.

21

In this case, repetitions $(x = y)$ are authorized and the order is counted.
Lets start.

As $x < z$ and $y < z$, $z > 2$.

- With $z = 2$, only one possibilities $(x;y) = (1;1)$.
- ª With $z = 3$, $(x;y) = (1;1)$, $(1;2)$, $(2;1)$, $(2;2)$.

With $z = 4$, $(x;y) = (1;1)$, $(1;2)$, $(1;3)$, $(2;1)$, $(2;2)$, $(2;3)$, $(3;1)$, $(3;2)$, $(3;3)$: $3\times 3 = 3^2$ possibilities.

…

At each level i, for $i = 1$ to 64, we have i^2 possibilities.

As a result the number of ordered triples is $1^2 + 2^2 + \ldots + 64^2 = \frac{64\times 65\times(2\times 64+1)}{6} = 89\,440$.

Answer: 89440.

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22 ◊

As $a_{n+1} = \dfrac{a_{n-1}}{1 + n\, a_{n-1}\, a_n}$, $\dfrac{1}{a_{n+1}} = \dfrac{1}{a_{n-1}} + n\, a_n$ so $\dfrac{1}{a_{n+1}\, a_n} = \dfrac{1}{a_{n+1}} \times \dfrac{1}{a_n} = \dfrac{1}{a_{n-1}\, a_n} + n$ then $\dfrac{1}{a_{n+1}\, a_n} - \dfrac{1}{a_n\, a_{n-1}} = n$.

We write $\dfrac{1}{a_{199}\, a_{200}} = \left(\dfrac{1}{a_{200}\, a_{199}} - \dfrac{1}{a_{199}\, a_{198}}\right) + \left(\dfrac{1}{a_{199}\, a_{198}} - \dfrac{1}{a_{198}\, a_{197}}\right) + \ldots + \left(\dfrac{1}{a_2\, a_1} - \dfrac{1}{a_1\, a_0}\right) + \dfrac{1}{a_1\, a_0}$ and so $\dfrac{1}{a_{199}\, a_{200}} = 199 + 198 + \ldots + 1 + 1 = \dfrac{199 \times 200}{2} + 1 = 19\,901$.

Answer: 19901.

23

We have all the lengths of the sides of the triangle *ABC* so it is completely defined.

The cosines law (Al-Kashi's Formula or Generalized Pythagorean Theorem) gives: $BC^2 = AB^2 + AC^2 - 2\, AB \times AC \cos(A)$ so $\cos(A) = \dfrac{-BC^2 + AC^2 + AB^2}{2\, AB\, AC} = -\dfrac{11}{25}$.

The Sines law (or area's law) gives $\mathcal{A}_{ABC} = \frac{1}{2} AB \times AC \times \sin(A) = 25 \sin(A)$ and $\mathcal{A}_{APQ} = \frac{1}{2} AQ \times AP \times \sin(A)$ so $\dfrac{\mathcal{A}_{APQ}}{\mathcal{A}_{ABC}} = \dfrac{1}{4}$ gives $\dfrac{\frac{1}{2} AP \times AQ}{25} = \dfrac{1}{4}$ and then $AP \times AQ = \dfrac{25}{2}$.

The cosines law yields to $PQ^2 = AP^2 + AQ^2 - 2\, AP \times AQ \times \cos(A)$ so $PQ^2 = AP^2 + \left(\dfrac{25}{AP}\right)^2 - 2 \times \dfrac{25}{2} \times \left(\dfrac{-11}{25}\right) = AP^2 + \dfrac{625}{4\, AP^2} + 11$.

Lets note that $\left(x + \dfrac{25}{2x}\right)^2 = x^2 + 2\, x \times \dfrac{25}{2x} + \dfrac{625}{4x^2} = x^2 + \dfrac{625}{4x^2} + 25 \geq 0$ (a square is positive) for all real x so $AP^2 + \dfrac{625}{4\, AP^2} \geq 25$ and then $PQ^2 \geq 25 + 11$ so $PQ^2 \geq 36$ and then $PQ \geq 6$.

Answer: 6.

24

As $x + y + z = 9$ then $x + y = 9 - z$ and as $x\, y + y\, z + x\, z = 24$ then $x\, y = 24 - z(x + y) = 24 - z(9 - z) = 24 - 9\, z + z^2$.

So $\begin{cases} x + y = 9 - z \\ x\, y = 24 - 9\, z + z^2 \end{cases}$. The real numbers x and y are thus the roots of the quadratic equation $X^2 - (9 - z)\, X + (z^2 - 9\, z + 24) = 0$.

Therefore, the maximum z possible is the maximum z such as this equation have real solutions.

The discriminant of this equation is $\Delta = (z - 9)^2 - 4\times(z^2 - 9\,z + 24) = -3\,z^2 + 18\,z - 15$.
This equation has real solutions if and only if $\Delta \geq 0$ so if and only if $z^2 - 6\,z + 5 \leq 0$.
As $z^2 - 6\,z + 5 = (z - 3)^2 - 4 = (z - 1)\,(z - 5)$, we deduce that the maximum value for z is $z = 5$.

Answer: 5.

25

Placing the six 1's, it leaves 7 gasps (one before, 5 in between and one after).
So we have to count the different ways of placing six 0's alone, two 0's and four alone, two times two 0's and 2 alone and 3 times two 0's.

Counting the ways to place the six 0's, is equivalent to count the number of words of length 7 with 6 letters identical and one blank.
There are $\dfrac{7!}{6!} = 7$ possibilities.

Counting the ways to place two 0's and four 0's alone is equivalent to count the number of words of length 7 with three different letters, one not repeated, one repeated 4 times and one repeated 2 times (2 blanks).
There are $\dfrac{7!}{4!\;\;2!} = 105$ possibilities.

Counting the ways to place two times two 0's and two 0's alone is equivalent to count the number of words of length 7 with three different letters, one repeated 2 times, one repeated 2 times and one repeated 3 times (3 blanks).
There are $\dfrac{7!}{2!\;\;2!\;\;3!} = 210$ possibilities.

Counting the ways to place three times two 0's is equivalent to count the number of words of length 7 with 2 different letters, one repeated 2 times, one repeated 3 times and one repeated 4 times (4 blanks).
There are $\dfrac{7!}{3!\;\;4!} = 35$ possibilities.

So we finally have 210 + 35 + 105 + 7 = 357 possibilities.

Answer: 357.

26

$$\frac{\cos(100\,°)}{1 - 4\sin(25\,°)\cos(25\,°)\cos(50\,°)} = .$$

$$\frac{\cos^2(50\,°) - \sin(50\,°)}{1 - 2\sin(50\,°)\cos(50\,°)} = \frac{(\cos(50\,°) - \sin(50\,°))\,(\cos(50\,°) + \sin(50\,°))}{(\cos(50\,°) - \sin(50\,°))^2} = \frac{\cos(50\,°) + \sin(50\,°)}{\cos(50\,°) - \sin(50\,°)}$$

In fact $2\sin(a)\cos(a) = \sin(2\,a)$ and $(\cos(a) - \sin(a))^2 = \cos^2(a) - 2\cos(a)\sin(a) + \sin^2(a) = 1 - 2\cos(a)\sin(a)$.

$$\frac{\cos(100\,°)}{1 - 4\sin(25\,°)\cos(25\,°)\cos(50\,°)} = \frac{\cos(50\,°)}{\cos(50\,°)} \times \frac{1 + \tan(50\,°)}{1 - \tan(50\,°)} = \frac{\tan(45\,°) + \tan(50\,°)}{1 - \tan(45\,°)\tan(50\,°)} = \tan(45\,° + 50\,°) = \tan(95\,°).$$

In fact $\tan(45\,°) = 1$ and $\tan(a + b) = \dfrac{\tan(a) + \tan(b)}{1 - \tan(a) + \tan(b)}$.

So $x = 95$.

Answer: 95.

27

$\mathrm{Log}_{\frac{x}{9}}\left(\frac{x^2}{3}\right) < 6 + \mathrm{Log}_3\left(\frac{9}{x}\right)$ can be written $\dfrac{\ln\left(\frac{x^2}{9}\right)}{\ln\left(\frac{x}{9}\right)} < 6 + \dfrac{\ln\left(\frac{9}{x}\right)}{\ln(3)}$ then $\dfrac{2\ln(x) - \ln(3)}{\ln(x) - 2\ln(3)} < 6 + \dfrac{2\ln(3) - \ln(x)}{\ln(3)}$ that gives $\dfrac{2\frac{\ln(x)}{\ln(3)} - 1}{\frac{\ln(x)}{\ln(3)} - 2} < 6 + 2 - \dfrac{\ln(x)}{\ln(3)}$.

Setting $u = \dfrac{\ln(x)}{\ln(3)}$, u verifies the inequation $\dfrac{2u-1}{u-2} < 8 - u$ which is equivalent to $\dfrac{2u-1}{u-2} + (u-8) < 0$ so to $\dfrac{u^2 - 8u + 15}{u-2} < 0$.

Yet $\dfrac{u^2 - 8u + 15}{u-2} = \dfrac{(u-3)(u-5)}{u-2}$ and using a table of signs, it follows that $\dfrac{u^2 - 8u + 15}{u-2} < 0$ if and only if $u < 2$ or $3 < u < 5$.

Thus:

- $\ln(x) < 2\ln(3)$ so $\ln(x) < \ln(9)$ and $0 < x < 9$: 8 integers.
- $3\ln(3) < \ln(x) < 5\ln(3)$ so $\ln(27) < \ln(x) < \ln(243)$ and $27 < x < 243$: $242 - 28 = 214$ integers.

Finally there are $214 + 9 = 223$ integers solutions.

Answer: 223.

28

We have for all positive integer p:

$$\frac{1}{p\sqrt{p+2} + (p+2)\sqrt{p}} = \frac{1}{\sqrt{p}\sqrt{p+2}} \times \frac{1}{\sqrt{p} + \sqrt{p+2}},$$

$$\frac{1}{p\sqrt{p+2} + (p+2)\sqrt{p}} = \frac{1}{\sqrt{p}\sqrt{p+2}} \times \frac{\sqrt{p+2} - \sqrt{p}}{2},$$

$$\frac{1}{p\sqrt{p+2} + (p+2)\sqrt{p}} = \frac{1}{2}\left(\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{p+2}}\right).$$

Thus $\dfrac{1}{9\sqrt{11} + 11\sqrt{9}} + \dfrac{1}{11\sqrt{13} + 13\sqrt{11}} + \ldots + \dfrac{1}{n\sqrt{n+2} + (n+2)\sqrt{n}} =$

$$\frac{1}{2}\left(\left(\frac{1}{\sqrt{9}} - \frac{1}{\sqrt{11}}\right) + \left(\frac{1}{\sqrt{11}} - \frac{1}{\sqrt{13}}\right) + \ldots + \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+2}}\right)\right)$$

So $\dfrac{1}{9\sqrt{11} + 11\sqrt{9}} + \dfrac{1}{11\sqrt{13} + 13\sqrt{11}} + \ldots + \dfrac{1}{n\sqrt{n+2} + (n+2)\sqrt{n}} = \dfrac{1}{2}\left(\dfrac{1}{3} - \dfrac{1}{\sqrt{n+2}}\right)$.

It follows $\dfrac{1}{2}\left(\dfrac{1}{3} - \dfrac{1}{\sqrt{n+2}}\right) = \dfrac{1}{9}$ so $\dfrac{1}{\sqrt{n+2}} = \dfrac{1}{9}$ then $n + 2 = 81$.

Finally $n = 79$.

Answer: 79.

29

As E is the midpoint of $[AD]$, $a_{CDE} = \frac{1}{2} a_{ACD}$ and $a_{ACD} = \frac{1}{2} a_{ABCD}$ so $a_{CDE} = \frac{1}{4} a_{ABCD}$.

As F is the midpoint of $[CE]$, then $a_{DFC} = \frac{1}{2} a_{CDE} = \frac{1}{8} a_{ABCD}$.

We also have $a_{BCE} = \frac{1}{2} a_{ABCD}$ and $a_{BCF} = \frac{1}{2} a_{BCE} = \frac{1}{4} a_{ABCD}$.

Noticing that $a_{BDF} = a_{BCD} - a_{BCF} - a_{DFC}$, it follows that $12 = \frac{1}{2} a_{ABCD} - \frac{1}{4} a_{ABCD} - \frac{1}{8} a_{ABCD} = \frac{1}{8} a_{ABCD}$.

Then $a_{ABCD} = 8 \times 12 = 96$

Answer: 96.

30 ◊

As every digit is at least 2 times in the six-digit number, such a six-digit number can only have 3 distinct digits.
There are 3 cases: only 1 digit, two distinct digits and three distinct digits.

First case: it is clear that there are 9 possibilities with only 1 digit repeated 6 times (0 is excluded as 000000=0 is a 1-digit number).

Second case:
We distinguish two cases: without 0 or with 0.
In fact such a number can not begin by 0.

- with 2 distinct non-zero digits: *a* can appear 2 times (4 *b*'s), 3 times (3 *b*'s) or 4 times (2 *b*'s .
  Lets call them *a* and *b*. Then:
  - *a* appears 2 times:
    The number of numbers with 2 digits *a* and 4 digits *b* is the number of combinations of 2 between 6 that is given by $\binom{6}{2} = \frac{6!}{2!\ \ 4!} = 15$. ($6!$ gives all the possibilities when the digits are taken distinct and as 1 is repeated 2 times, each sequence is counted 2 times and as the other is repeated 4 times each remaining sequence is counted $4\times3\times2\times1 = 4!$ which is the numbers of possibilities to permute 4 letters).
  - *a* appears 3 times: we count $\binom{6}{3} = \frac{6!}{3!\ \ 3!} = 20$ possibilities.
  - *a* appears 4 times: we count $\binom{6}{4} = 15$ possibilities.
- with at least 2 zero: the first digit is a, $a \neq 0$.
  There are 5 digits left: 3 *a*'s 2 0's, 2 *a*'s and 3 0's or 1 *a* and 4 0's.
  We count $\binom{5}{2} + \binom{5}{3} + \binom{5}{4} = 25$ possibilities.

Lets now observe there are $\frac{9\times8}{2} = 36$ (the order is not counted) ways to choose two distinct non-zero digits and 9 ways to choose a non-zero digit, so the total of 6-digit numbers as defined with two distinct digits is $36\times50 + 9\times25 = 2025$.

Third case:
Identically we distinguish three distinct non-zero digits and two distinct non-zero digits plus 1 zero.

• with three distinct non-zero digits, there are $\binom{6}{2}$ ways to place two identical digits in 6 places and then for each precedent case there are 4 places left and so $\binom{4}{2}$ ways to place 2 identical digits in the 4 places left. The last two places are defined by the precedent cases.

So we count $\binom{6}{2}\times\binom{4}{2} = 90$ ways to place three pairs of three non-zero digits.

• with a zero, the first place being taken by one of the non-zero digit, there are 5 places left.

2 places are taken by the 2 zero's: $\binom{5}{2}$ and there are 3 places left so each precedent case is multiply by $\binom{3}{2}$ possibilities of placing the one of the non-zero digit which has not been placed at the first place.

As we do not know which one of the non-zero digit has been taken, we then count $2\times\binom{5}{2}\times\binom{3}{2} = 60$ ways.

For three non-zero digits, there are $\frac{9\times 8\times 7}{2\times 3} = 84$ ways of choosing three distinct digits and for two non-zero digits there are 48 ways to choose two distinct digits.
It follows that there are $84\times 90 + 36\times 60 = 9720$ possibilities.

In total there are 9 + 2025 + 9720 = 11 754 different numbers.

Answer: 11754.

31

Note that $945 = 27\times 35$.

The positive integers x and y are such that $27x + 35y \leq 945$ so $y \leq \frac{945}{35} - \frac{27}{35}x$.

Thus $xy \leq \frac{27}{35}\left(35x - x^2\right)$ and finally $xy \leq \frac{27}{35}\left(\left(\frac{35}{2}\right)^2 - \left(x - \frac{35}{2}\right)^2\right)$.

The maximum of $\frac{27}{35}\left(\left(\frac{35}{2}\right)^2 - \left(x - \frac{35}{2}\right)^2\right)$ is for $x = \frac{35}{2} \notin \mathbb{N}$equal to $\frac{27\times 35}{4} = 236, 25$.

Then the possible values are symmetric regarding $\frac{35}{2}$.

The problem is then to find an integer couple also satisfying $27x + 35y \leq 945$.

If $x \leq 15$ or $x \geq 20$, $\left(x - \frac{35}{2}\right)^2 \geq \frac{25}{4}$ and so $\frac{27}{35}\left(\left(\frac{35}{2}\right)^2 - \left(x - \frac{35}{2}\right)^2\right) \leq \frac{27}{35}\times\left(\frac{1200}{4}\right) \leq 231, 4$.

Lets look for the remaining integers values for x: 16, 17, 18 or 19.

• $x = 16$ gives $y \leq 27 - \frac{27}{35}\times 16$ so $y \leq 14$ and $xy \leq 224$
• $x = 17$ gives $y \leq 13$ so $xy \leq 221$
• $x = 18$ gives $y \leq 13$ so $xy \leq 234$
• $x = 19$ gives $y \leq 12$ so $xy \leq 228$.

In conclusion the maximum value possible is reached for $(x; y) = (18; 13)$ and is equal to 234.

Answer: 234.

32

Here we need the binomial formula $(x+1)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$.

$\left(1+x^5+x^7+x^9\right)^{16} = \left(1+x^5\left(1+x^2+x^4\right)\right)^{16} = \sum_{i=0}^{16} \binom{16}{i} x^{5i}\left(1+x^2+x^4\right)^i$.

If $i \geq 6$, then $5i \geq 30$ and so there are no way to get x^{29}.

If $i \leq 3$, so $5i \leq 15$ and as $4i \leq 12$ then at the most we reach $x^{15+12} = x^{27}$ expanding $x^{5i}(1+x^2+x^4)^i$.
It remains $i = 4$ and $i = 5$.

If $i = 4$, then $5i = 20$ and $x^{20}(1+x^2+x^4)^4$ only have even exponent: we can not reach x^{29}.

Finally we only have to study the expansion of $\binom{16}{5} x^{25}(1+x^2+x^4)^5$.

We have $\left(1+x^2+x^4\right)^5 = \left(1+x^2\left(1+x^2\right)\right)^5 = \sum_{i=0}^{5} \binom{5}{i} x^{2i}\left(1+x^2\right)^i$.

Then $\binom{16}{5} x^{25}\left(1+x^2+x^4\right)^5 = \binom{16}{5} x^{25} \sum_{i=0}^{5} \binom{5}{i} x^{2i}\left(1+x^2\right)^i$ and so the coefficient of x^{29} is given by the coefficient of x^4 in the expansion of $\sum_{i=0}^{5} \binom{5}{i} x^{2i}\left(1+x^2\right)^i$.

If $i \geq 3$, we have at least x^6 and if $i = 0$, no exponent at all.
So $i = 1$ or $i = 2$.

If $i = 1$, $x^{2\times 1}(1+x^2)^1 = x^2 + x^4$ and so the coefficient of x^4 is $\binom{5}{1}$.

If $i = 2$, $x^4(1+x^2)^2 = x^4 + 2x^6 + x^8$ so the coefficient of x^4 is $\binom{5}{2}$.

Finally the coefficient of x^4 is $\binom{5}{1} + \binom{5}{2} = 15$ and so the coefficient of x^{29} is $\binom{16}{5} \times 15 = \frac{16!}{5!\ \ 11!} \times 15 = \frac{12\times 13\times 14\times 15\times 16}{1\times 2\times 3\times 4\times 5} \times 15 = 4368 \times 15 = 65\,520$.

Answer: 65520.

33

As $d_n$is a common divisor of a_n and a_{n+1} then d_n is also a divisor of $a_{n+1} - a_n$.

Yet $a_{n+1} - a_n = (n+1)^2 - 100 - n^2 + 100 = 2n + 1$.
Thus $d_n$divides $2n + 1$.

So it exists two integers k and k' such that $\begin{cases} n^2 + 100 = k\, d_n \\ 2n + 1 = k'\, d_n \end{cases}$.

Eliminating n^2 in the system drives to $2(n^2 + 100) - n\times(2n+1) = (2k - n k')\, d_n$ so $200 - n = K\, d_n$: d_n divides $200 - n$.

So identically $\begin{cases} 2n+1 = p\,d_n \\ 200-n = p'\,d_n \end{cases}$ which yields to $2\times(200-n)+(2n+1)=(2p'+p)\,d_n$ and so $401 = K'\,d_n$.

Finally d_n divides 401 so $1 \le d_n \le 401$.

As d_n must be the greatest, lets try to find $\{a_n; a_{n+1}\}$ such that 401 is their greatest common divisor.

We are then looking for an integer n such that $n^2+100 = 401\times k$ and $(n+1)^2+100 = 401\times k'$.
As seen already, $2n+1 = 401\,(k'-k)$.
Lets note that $401 = 2\times 200+1$ so $n = 200$ could be a solution.
In fact $a_{200} = 200^2+100 = 40\,100 = 401\times 100$ and $201^2+100 = 40\,501 = 401\times 101$.
So 401 is the greatest common divisor of a_{200} and a_{201}.
As for all n $d_n \le 401$ so the greatest common divisor of a_n and a_{n+1} for all positive integers n is 401.

Answer: 401.

34

Answer: 441.

35

Answer: 24.