Maths Lab: Wordings

Maths Lab: Elements of solution

Multiple Choice Questions

1

(A) 5

Notice that for all strictly positive integers n, $1+\frac{1}{n}=\frac{n+1}{n}$.

Thus $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\ldots\left(1+\frac{1}{2006}\right)\left(1+\frac{1}{2007}\right)=\frac{3}{2}\times\frac{4}{3}\times\ldots\times\frac{2007}{2006}\times\frac{2008}{2007}=\frac{2008}{2}=1004$.

The sum of the digits of the product is 5.

2

(C) $\frac{y}{x+y}$

The second sweet drawn is red if the first sweet drawn was red and the second sweet drawn is red or if the first sweet drawn was green and the second sweet drawn is red.

The first sweet drawn is red with the probability $\frac{y}{x+y}$.

Then the bag contains $x+y+10$ sweets whose $y+10$ are red.

The probability to draw a red sweet, the second time, assuming that the first sweet drawn was red, is then $\frac{y+10}{x+y+10}$.

Finally, the probability to draw a red sweet the second time and a red sweet the first time is given by $\frac{y}{x+y}\times\frac{y+10}{x+y+10}$.

The first sweet drawn is green with the probability $\frac{x}{x+y}$.

Then the bag contains $x+y+10$ sweets whose y are red.

The probability to draw a red sweet, the second time, assuming that the first sweet drawn was green, is then $\frac{y}{x+y+10}$.

Finally, the probability to draw a red sweet the second time and a red sweet the first time is given by $\frac{x}{x+y}\times\frac{y}{x+y+10}$.

Therefore, the probability to draw a red sweet the second time is given by $\frac{x}{x+y}\times\frac{y}{x+y+10}+\frac{y}{x+y}\times\frac{y+10}{x+y+10}=\frac{y(x+y+10)}{(x+y)(x+y+10)}=\frac{y}{x+y}$.

3

(A) 0

Notice that $\underbrace{999\ldots999}_{2008\text{'s }9} = 9\times\underbrace{111\ldots111}_{2008\text{'s }1}$ and $\underbrace{333\ldots333}_{2008\text{'s }3} = 3\times\underbrace{111\ldots111}_{2008\text{'s }1}$.

Or $\underbrace{111\ldots111}_{2008\text{'s }1}$ is divisible by 11: $\underbrace{111\ldots111}_{2008\text{'s }1} = 11\times 1\,\underbrace{000\ldots000}_{2006\text{'s }0} + 11\times 1\,\underbrace{000\ldots000}_{2004\text{'s }0} + \ldots + 11$.

Then $\underbrace{999\ldots999}_{2008\text{'s }9} = 9\times\underbrace{111\ldots111}_{2008\text{'s }1}$ and $\underbrace{333\ldots333}_{2008\text{'s }3} = 3\times\underbrace{111\ldots111}_{2008\text{'s }1}$ are divisible by 11.

Hence $(\underbrace{999\ldots999}_{2008\text{'s }9})^{2007} - (\underbrace{333\ldots333}_{2008\text{'s }3})^{2007} = (11\times k)^{2007} - (11\times k')^{2007} = 11^{2007}\times\left(k^{2007} - (k')^{2007}\right)$ is divisible by 11.

4

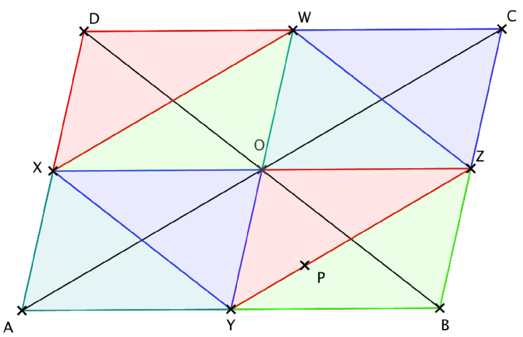
(D) 25%

It is clear that the quadrilateral $XYZW$ is a parallelogram (*Varignon's theorem: the quadrilateral formed when the midpoints of the sides of a convex quadrilateral are joined in order is a parallelogram*) and that wherever the point P is on the line YZ, the area of the triangle PXW is half the area of the parallelogram $XYZW$.
They share the same height.

As the center O of the parallelogram $ABCD$ is also the center of the parallelogram $XYZW$, we can affirm that the triangle OXW is the image of the triangle BYZ by translation of vector $\overrightarrow{BO}$. Then they have the same area.
The same for the triangle OYZ and DXW.

Using the translation of vector $\overrightarrow{AO}$, we obtain that the triangles AXY and OWZ have the same area and the triangles OXY and CWZ also.

Finally the area of the parallelogram $XYZW$ is half the area of the parallelogram $ABCD$.
The triangle XWP is the 25% of the area of the parallelogram $ABCD$.

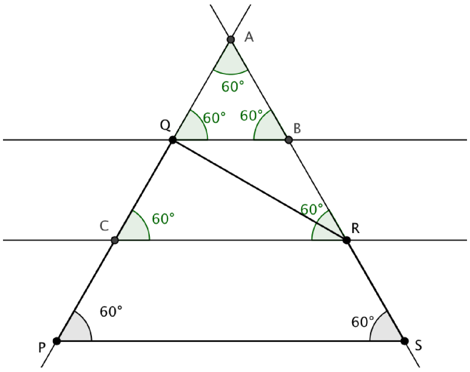


5

(E) 150°

As $\angle QPS = \angle RSP = 60°$, naming A the intersection of the lines PQ and SR, the triangle APS is equilateral.
Then as $AP = AS = PS = 60$ and as $PQ = 40$, $AQ = 20$ and as $SR = 20$, $AS = 40$.
Let BQ and CR be the parallel lines to PS through R and Q with B on AS and C on AP.
Then corresponding angles being equal, we deduce that the triangles AQB and ACR are equilateral with respective side length equal to 20 and 40.

Then $AQ = QC = \frac{1}{2}AC$.
Line QR is the median through the vertex R in the triangle ACR.
As ACR is equilateral, it is also the height and so it is perpendicular to the side AC.
We have $\angle PQR = 90°$.



We deduce $\angle CRQ = 30°$, then $\angle QRS = \angle QRC + \angle CRS = 30° + (180° - 60°) = 150°$.

6

(A) 14

We have $2007 = k \times N + 5$ then $k \times N = 2002$.
It follows that N is a divider of 2002.

Or $2002 = 2 \times 7 \times 11 \times 13$.
The list of the dividers of 2002 is {1; 2; 7; 11; 13; 14; 22; 26; 77; 91; 143; 154; 182; 286; 1001; 2002}.

But $N \geq 5$(in fact ortherwise, there can't be 5 soaps remaining). There are 14 possible values of N.

7

(B) 50199

Knowing the rules of multiplication, the last 2 digits of 7^n are given by the last digits of 7^{n-1} multiply by 7.
We have:
$7^1 = 7$, $7^2 = 49$, $7^3 = 243$, $7^4 = 7^3 \times 7 = (\ldots 43) \times 7 = \ldots 01$, $7^5 = (\ldots 01) \times 7 = \ldots 07$, $7^6 = (\ldots 07) \times 7 = \ldots 49$ and so on.
thus the last two digits of 7^n are 07; 49; 43; 01 beginning at $n = 1$.
They are repeating in loops of 4.

As $2007 = 4 \times 501 + 3$, the the sum $a_1 + a_2 + \ldots + a_{2007}$ is given by $(07 + 49 + 43 + 01) \times 501 + 07 + 43 + 49 = 50\,199$.

8

(D) 31

Knowing that the area of a triangle is given by $\mathcal{A} = b\,c \sin(\hat{A})$ where b et c are the length of the adjacent sides to the vertex A, and $\hat{A}$ the angle at the vertex A, we obtain:

$$\mathcal{A}_{ABO} = \frac{1}{2}\, OA \times OB \times \sin(135°)$$

$$\mathcal{A}_{ADO} = \frac{1}{2}\, OA \times OD \times \sin(45°)$$

$$\mathcal{A}_{BCO} = \frac{1}{2}\, OB \times OC \times \sin(45°)$$

$$\mathcal{A}_{CDO} = \frac{1}{2}\, OC \times OD \times \sin(135°).$$

As $\sin(45°) = \sin(135°) = \dfrac{\sqrt{2}}{2}$, Thus $\mathcal{A}_{ABCD} = \frac{1}{2}\times\frac{\sqrt{2}}{2}\times(OA\times OB + OA\times OD + OB\times OC + OC\times OD) = \frac{\sqrt{2}}{4}\,(OA\times(OB+OD) + OC\times(OB+OD))$,

As $OB + OD = BD$ and $OA + OC = AC$, $\mathcal{A}_{ABCD} = \frac{\sqrt{2}}{4}\,(OA\times BD + OC\times BD) = \frac{\sqrt{2}}{4}\,AC\times BD$.

Lets show that $AC\times BD = \dfrac{AB^2 + CD^2 - AD^2 - BC^2}{2\cos(45^\circ)}$.

Using the cosines law in a triangle:

$AB^2 = OA^2 + OB^2 - 2\,OA\times OB\cos(135^\circ) = OA^2 + OB^2 + 2\,OA\times OB\cos(45^\circ)$

$AD^2 = OA^2 + OD^2 - 2\,OA\times OD\cos(45^\circ)$

$BC^2 = OB^2 + OC^2 - 2\,OB\times OC\cos(45^\circ)$

$CD^2 = OC^2 + OD^2 - 2\,OC\times OD\cos(135^\circ) = OC^2 + OD^2 + 2\,OC\times OD\cos(45^\circ)$.

Thus:

$$AB^2 + CD^2 - AD^2 - BC^2 = \begin{array}{l} OA^2 + OB^2 + 2\,OA\times OB\cos(45^\circ) \\ +OC^2 + OD^2 + 2\,OC\times OD\cos(45^\circ) \\ -OA^2 - OD^2 - 2\,OA\times OD\cos(45^\circ) \\ -OB^2 + OC^2 - 2\,OB\times OC\cos(45^\circ) \end{array};$$

$AB^2 + CD^2 - AD^2 - BC^2 = 2\cos(45^\circ)\,(OA\times OB + OA\times OD + OB\times OC + OC\times OD) = 2\cos(45^\circ)\,AC\times BD$.

Therefore, $\mathcal{A}_{ABCD} = \dfrac{\sqrt{2}}{4}\times\dfrac{AB^2 + CD^2 - AD^2 - BC^2}{2\cos(45^\circ)} = \dfrac{1}{4}\left(AB^2 + CD^2 - AD^2 - BC^2\right)$.

In fact $\dfrac{\sqrt{2}}{4}\times\dfrac{1}{2\cos(45^\circ)} = \dfrac{\sqrt{2}}{8}\times\dfrac{1}{\frac{\sqrt{2}}{2}} = \dfrac{2}{8} = \dfrac{1}{4}$.

Finalement $\mathcal{A}_{ABCD} = \dfrac{1}{4}\left(10^2 + 8^2 - 6^2 - 2^2\right) = 31$.

Remarks:

We therefore demonstrate 2 formulas for the area of a quadrilateral:

$\mathcal{A} = \dfrac{1}{4}\left(b^2 + d^2 - a^2 - c^2\right)\tan(\theta)$, where a, b, c and d are the lengths of the sides of the quadrilateral and θ is the angle between the 2 diagonals, "looking at" the sides of length a or c.

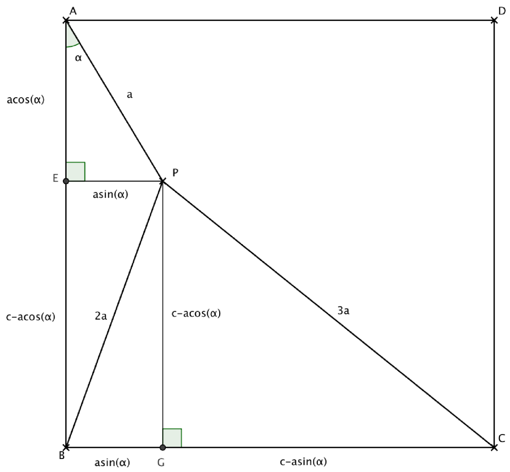
$\mathcal{A} = \dfrac{1}{2}\,p\,q\sin(\theta)$ where p and q are the lengths of the diagonals, and θ the angle formed by these two diagonals.

9

(C) 135º

Difficult.

Lets note c th length of the side of the square, $\alpha = \angle PAB$ and $x = \angle APC$.



The laws of cosines gives:
$AB^2 = PA^2 + PB^2 - 2\,PA \times PB \times \cos(\angle APB)$: $c^2 = a^2 + 4\,a^2 - 4\,a^2\cos(x) = 5\,a^2 - 4\,a^2\cos(x)$.

Thus $\cos(x) = \dfrac{5\,a^2 - c^2}{4\,a^2} = \dfrac{5 - \left(\frac{c}{a}\right)^2}{4}$.

So we have to find the ratio between c and a.
The difficulty lies here to link the 3 conditions: a, $2\,a$ and $3\,a$.
One must know that an angle is determinate knowing its cosinus and its sinus.

The law of cosines in triangle *APB* also gives:

$(2\,a)^2 = c^2 + a^2 - 2\,a\,c\cos(\alpha)$ so $\cos(\alpha) = \dfrac{c^2 - 3\,a^2}{2\,a\,c}$.

Introducing the angle $\alpha = \angle PAC$, and projecting P on sides *AB* and *BC*, we built rectangle triangles, whose sides can be expressed using a, c and α.
The Pythagoras theorem in triangle *PGC* gives:
$(3\,a)^2 = (c - a\cos(\alpha))^2 + (c - a\sin(\alpha))^2$.
$9\,a^2 = c^2 - 2\,a\,c\cos(\alpha) + a^2\cos^2(\alpha) + c^2 - 2\,a\,c\sin(\alpha) + a^2\sin^2(\alpha)$.

Knowing that $\cos^2 + \sin^2 = 1$, it gives $\sin(\alpha) = \dfrac{2\,c^2 - 8\,a^2}{2\,a\,c} - \cos(\alpha) = \dfrac{2\,c^2 - 8\,a^2}{2\,a\,c} - \dfrac{c^2 - 3\,a^2}{2\,a\,c} = \dfrac{c^2 - 5\,a^2}{2\,a\,c}$.

The angle α is then given by $\begin{cases} \cos(\alpha) = \dfrac{c^2 - 3\,a^2}{2\,a\,c} \\ \sin(\alpha) = \dfrac{c^2 - 5\,a^2}{2\,a\,c} \end{cases}$.

As again, $\cos^2(\alpha) + \sin^2(\alpha) = 1$, numbers a and c verify $\left(\frac{c^2 - 3\,a^2}{2\,a\,c}\right)^2 + \left(\frac{c^2 - 5\,a^2}{2\,a\,c}\right)^2 = 1$, which gives $(c^2 - 3\,a^2)^2 + (c^2 - 5\,a^2) = 4\,a^2\,c^2$ then $2\,c^4 - 16\,a^2\,c^2 + 34\,a^4 = 4\,a^2\,c^2$ and finally $c^4 - 10\,a^2\,c^2 + 17\,a^4 = 0$ or $\left(\dfrac{c}{a}\right)^4 - 10\left(\dfrac{c}{a}\right)^2 + 17 = 0$.

Note:
We call such equation biquadratic equations.

Lets note $X = \left(\dfrac{c}{a}\right)^2$, then X is solution of the quadratic equation $X^2 - 10\,X + 17 = 0$.

The discriminant of this equation is $\Delta = (10)^2 - 4 \times 17 = 32 > 0$.

The equation has 2 solutions: $X_1 = \dfrac{-(-10) - \sqrt{32}}{2} = 5 - 2\sqrt{2} < 1$ and $X_2 = 5 + 2\sqrt{2} > 1$.

Thus $\left(\dfrac{c}{a}\right)^2 = 5 - 2\sqrt{2}$ or $\left(\dfrac{c}{a}\right)^2 = 5 + 2\sqrt{2}$.

As $c > a$, $\frac{c}{a} > 1$ and then the only possibility is $\left(\frac{c}{a}\right)^2 = 5 + 2\sqrt{2}$.

Finally as $\cos(x) = \dfrac{5 - \left(\frac{c}{a}\right)^2}{4}$, $\cos(x) = \dfrac{5 - \left(5 + 2\sqrt{2}\right)}{4} = -\dfrac{\sqrt{2}}{2}$.

It falls $x = 135\,°$.

Note:

For all quadratic equations with the form $a x^2 + b x + c = 0$, $a \neq 0$, $x \in \mathbb{R}$, we know that:

The equation has:

- 2 distincts solutions if and only if $\Delta = b^2 - 4 a c > 0$ which are $x_1 = \dfrac{-b - \sqrt{\Delta}}{2a}$ and $x_2 = \dfrac{-b + \sqrt{\Delta}}{2a}$;
- 1 solution if and only if $\Delta = 0$ which is $x_0 = -\dfrac{b}{2a}$;
- no solution if and only if $\Delta < 0$.

Lets also remember that $a x^2 + b x + c = a\left[\left(x - \left(-\dfrac{b}{2a}\right)\right)^2 - \dfrac{b^2 - 4ac}{4a}\right]$ (canonical form), and in the good cases $a x^2 + b x + c = a(x - x_1)(x - x_2)$ (factored form for $\Delta > 0$) or $a x^2 + b x + c = a(x - x_0)^2$ (factored form for $\Delta = 0$).

Short Questions

10

(E) 7

We have $n^2 - 12n + 27 = (n - 3)(n - 9)$.

A number is a prime number if and only if his only divisors are 1 and himself.

Therefore, $n^2 - 12n + 27$ is a prime number if and only if $n - 3 = 1$ or -1 or $n - 9 = 1$ or -1 and $(n - 3)(n - 9) > 0$.

Then $n = 4$ (impossible because $4 - 3 > 0$ and $4 - 9 < 0$) or $n = 2$ or $n = 8$(impossible because $7 - 9 < 0$ and $7 - 3 > 0$) or $n = 10$.

For $n = 2$, $2^2 - 12\times 2 + 27 = 7$ and for $n = 10$, $10^2 - 12\times 10 + 27 = 7$.

7 is the largest prime value possible.

11

Informations:

$\log_a$ for all strictly positive real a, $a \neq 1$, called logarithm of base a is the function such as $y = \log_a(x) \Longleftrightarrow x = a^y$.

We can then notice $\log_a(x) = 0$ if and only if $x = 1$ and $\log_a(a) = 1$.

A fundamental property is: $\log_a(x\,y) = \log_a(x) + \log_a(y)$.

A consequence is $\log_a(x^n) = n\log_a(x)$. In particular $\log_a(a^n) = n$.

We have:

- $\log_2[\log_3[\log_4(a)]] = 0$ then $\log_3[\log_4(a)] = 1$ and $\log_4(a) = 3$. Finally $a = 4^3 = 64$.
- $\log_3[\log_4[\log_2(b)]] = 0$ then $\log_4[\log_2(b)] = 1$ and $\log_2(b) = 4$. Finally $b = 2^4 = 16$.
- $\log_4[\log_2[\log_3(c)]] = 0$ then $\log_2[\log_3(b)] = 1$ and $\log_3(b) = 2$. Finally $c = 3^2 = 9$.

We obtain $a + b + c = 89$.

12

The unit digit of a product is given by the product of the unit digit of the factors.

Therefore:

• the unit digit of 17^{17} is the unit digit of 7^{17}.
Or $7^1 = 7$, $7^2 = \ldots 9$ $(7\times 7 = 49)$, $7^3 = \ldots 3$ $(9\times 7 = 63)$, $7^4 = \ldots 1$ $(3\times 7 = 21)$, $7^5 = \ldots 7$ $(1\times 7 = 7)$: we have a loop of length 4.
As $17 = 4\times 4 + 1$, we deduce that the unit digit of 17^{17} is 7.
• the unit digit of 19^{19} is the unit digit of 9^{19}.
Or $9^1 = 9$, $9^2 = \ldots 1$ $(9\times 9 = 81)$, $9^3 = \ldots 9$ $(1\times 9 = 9)$, $7^4 = \ldots 1$ $(9\times 9 = 81)$: we have a loop of length 2.
As $19 = 18\times 2 + 1$, we deduce that the unit digit of 19^{19} is 9.
• the unit digit of 23^{23} is the unit digit of 3^{23}.
Or $3^1 = 3$, $3^2 = \ldots 9$ $(3\times 3 = 9)$, $3^3 = \ldots 7$ $(9\times 3 = 27)$, $3^4 = \ldots 1$ $(7\times 3 = 21)$, $3^5 = \ldots 3$ $(1\times 3 = 3)$: we have a loop of 4.
As $23 = 5\times 4 + 3$, we deduce that the unit digit of 23^{23} is 7 (the third place in the loop).

Thus the unit digit of $17^{17}\times 19^{19}\times 23^{23}$ is the unit digit of the product $7\times 9\times 7 = \ldots 1$.
1 is the unit digit of $17^{17}\times 19^{19}\times 23^{23}$.

13

$(x + y)^2 = x^2 + 2\,x\,y + y^2$ thus $x^2 + y^2 = (x + y)^2 - 2\,x\,y = 12^2 - 2\times 50 = 44$.

14

Informations:
The notion of exponant can be generalized to whatever real applied to at least all positive real.
For a, b reals, and x, y positive reals:
• $x^a = 0$ if and only if $x = 0$ and $a \neq 0$.
• $x^a = 1$ if and only if $x = 1$.
• $x^a\,y^a = (x\,y)^a$.
• $x^a\,x^b = x^{a+b}$.
• $(x^a)^b = x^{a\,b}$. In particular $(x^a)^{\frac{1}{a}} = x$ and $\left(x^{\frac{1}{a}}\right)^a = x$ for all real $a \neq 0$.

Let's extract the exponant.
A common way to extract an exponent is to use the logarithm: $\log_a(x^n) = n\log_a(x)$.
As $(21.4)^a = (0.00214)^b = 100$ and as $0.00214 = 21.4\times 10^{-4}$, it looks interesting to use $\log_{10}$ as $\log_{10}(10^n) = n\log_{10}(10) = n$.
(it is one of the most interesting property of $\log_{10}$).

$\log_{10}((21.4)^a) = a\log_{10}(21.4)$ or $(21.4)^a = 100$ then $a\log_{10}(21.4) = \log_{10}\left(10^2\right) = 2$: $a = \dfrac{2}{\log(21.4)}$.

$\log_{10}\left((0.00214)^b\right) = b\log_{10}\left(21.4\times 10^{-4}\right) = b\left(\log_{10}(21.4) + \log_{10}\left(10^{-4}\right)\right) = b\left(\log_{10}(21.4) - 4\right)$.

As $(0.00214)^b = 100$, we obtain $b\left(\log_{10}(21.4) - 4\right) = 2$ then $b = \dfrac{2}{\log_{10}(21.4) - 4}$.

Finally we have:

$$\frac{1}{a} - \frac{1}{b} = \frac{\log_{10}(21.4)}{2} - \frac{\log_{10}(21.4) - 4}{2} = \frac{4}{2} = 2.$$

15

Notes:

$\sin(-\alpha) = -\sin(\alpha)$	$\cos(-\alpha) = \cos(\alpha)$
$\sin(180^\circ - \alpha) = \sin(\alpha)$	$\cos(180^\circ - \alpha) = -\cos(\alpha)$

$\sin(180° + \alpha) = -\sin(\alpha)$ $\cos(180° + \alpha) = -\cos(\alpha)$
$\sin(360° - \alpha) = -\sin(\alpha)$ $\cos(360° - \alpha) = -\cos(\alpha)$

$\sin(90° - \alpha) = \cos(\alpha)$ $\cos(90° - \alpha) = \sin(\alpha)$
$\sin(90° + \alpha) = \cos(\alpha)$ $\cos(90° + \alpha) = -\sin(\alpha)$

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
$\sin(\alpha + \beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$
$\sin(\alpha - \beta) = \cos(\alpha)\sin(\beta) - \sin(\alpha)\cos(\beta)$

Therefore:
$100\,(\sin(253°)\sin(313°) + \sin(163°)\sin(223°)) = 100\,((-\sin(73°))\,(-\sin(47°)) + \sin(17°)\,(-\sin(43°)))$
$100\,(\sin(253°)\sin(313°) + \sin(163°)\sin(223°)) = 100\,(\sin(73°)\sin(47°) - \sin(17°)\sin(43°))$
$100\,(\sin(253°)\sin(313°) + \sin(163°)\sin(223°)) = 100\,(\cos(17°)\sin(47°) - \sin(17°)\cos(47°))$
$100\,(\sin(253°)\sin(313°) + \sin(163°)\sin(223°)) = 100\sin(47° - 17°)$
$100\,(\sin(253°)\sin(313°) + \sin(163°)\sin(223°)) = 100\sin(30°)$
$100\,(\sin(253°)\sin(313°) + \sin(163°)\sin(223°)) = 100\times\frac{1}{2}$
$100\,(\sin(253°)\sin(313°) + \sin(163°)\sin(223°)) = 50.$

16

Suppose all the letters distincts.
Then we have 4 vowels and 7 consonants.
As the 4 vowels are the first 4 letters, we can treat the problem as if we have 2 words: one with the vowels and one with the consonants.
We then have $4\times3\times2\times1 = 24$ possibilities of words with 4 distinct vowels and $7\times6\times5\times4\times3\times2\times1 = 5040$ possibilities of words for 7 distincts consonants.

As 2 vowels are identical, each word is counted 2 times: we then have $\frac{24}{2} = 12$ distincts words of 4 vowels, with 2 vowels identical.

As 2 pairs of consonants are identical, each word is counted $2\times2 = 4$ times.
We then have $\frac{5040}{4} = 1260$ words of 7 consonants, with two pairs of identical consonants.

Finally we have $12\times1260 = 15\,120$ distinct arrangements.

17

As $a + c = 2\,b$, then $a + c$ is even.
Therefore or a, c are both even or a, c are both odd.
There are 100 odd numbers and 100 even numbers in S.

There are 100×99 ways to choose two distinct numbers among 100 considering the order, and then $\frac{100\times99}{2}$ ways to choose two distinct numbers among 100 not considering the order.

Considering that whatever choice of the numbers a and c in S, having the same parity, it exist a number b in S as $a + c = 2\,b$ (in fact $a + c \leq 400$ and $a + c$ is even so the half of $a + c$ (b is an integer ≤ 200) and conidering that a and c being choosen, b is determined, there are for each parity of a and c, $\frac{100\times99}{2}$ choices possibles.

Finally there are $2\times\frac{100\times99}{2} = 9900$ choices possibles.

18

For all real x and all odd integer n, $x^n + 1 = (x + 1)\,(x^{n-1} - x^{n-2} + \ldots - x + 1)$ (The factor theorem).
In fact $(x + 1)\,(x^{n-1} - x^{n-2} + \ldots - x + 1) = (x^n - x^{n-1} + x^{n-2} \ldots - x^2 + x) + (x^{n-1} - x^{n-2} + \ldots + x^2 - x + 1)$: the terms simplify 2 by 2 except for x^n and 1.

So $2^{55} + 1 = (2^5)^{11} + 1 = (2^5 + 1)\times N$ avec $N = \big((2^5)^{10} - (2^5)^9 + (2^5)^8 - \ldots + (2^5)^2 - 2^5 + 1$.

Or $2^5 + 1 = 32 + 1 = 33$.
33 divide $2^{55} + 1$.
The remainder is 0.

19

Let n et m be these 2 numbers with $n > m$.
We have $n - m = 58$.
We also have $n^2 - m^2 = (n - m)\,(n + m) = 58\,(n + m)$ is a multiple of 100, as the 2 last digits of the squares of these two numbers are the same, the last 2 digits of the difference between the 2 squares are 00.

Then $58\,(n + m) = 100\,k$ d'où $29\,(n + m) = 50\,k$.
Since 29 and 50 are coprime, therefore 50 divide $n + m$: it exists an integer positive p as $n + m = 50\,p$.

Keeping in mind that n and m are 2-digits numbers, $n \le 99$ and $m \le 99$ so $n + m < 200$.
Though, $n + m = 50$ or $n + m = 100$ or $n + m = 150$.

If $n - m = 58$ and $n + m = 50$, $2\,n = 108$ so $n = 54$ and $m = -4 < 0$: impossible.
If $n - m = 58$ and $n + m = 100$, then $2\,n = 158$ so $n = 79$ and $m = 21$.
If $n - m = 58$ and $n + m = 150$, then $2\,n = 208$ so $n = 104 > 99$: impossible.

Only one solution remains possible: $n = 79$ and $m = 21$.
The smaller number is 21.

20

Notes:

$\sin(2\,a) = 2\cos(a)\sin(a)$ $\qquad$ $\cos(2\,a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$
$\sin(90^\circ - \alpha) = \cos(\alpha)$
$\sin(30^\circ) = \dfrac{1}{2}$

$$256\sin(10^\circ)\sin(30^\circ)\sin(50^\circ)\sin(70^\circ) = 256\sin(10^\circ)\times\frac{1}{2}\times\cos(40^\circ)\times\cos(20^\circ)$$
$$256\sin(10^\circ)\sin(30^\circ)\sin(50^\circ)\sin(70^\circ) = 128\sin(10^\circ)\cos(20^\circ)\cos(40^\circ)$$
$$256\sin(10^\circ)\sin(30^\circ)\sin(50^\circ)\sin(70^\circ) = \frac{128\sin(10^\circ)\times 2\cos(10^\circ)\times\cos(20^\circ)\cos(40^\circ)}{2\cos(10^\circ)}$$
$$256\sin(10^\circ)\sin(30^\circ)\sin(50^\circ)\sin(70^\circ) = \frac{64\sin(20^\circ)\times\cos(20^\circ)\cos(40^\circ)}{\cos(10^\circ)}$$
$$256\sin(10^\circ)\sin(30^\circ)\sin(50^\circ)\sin(70^\circ) = \frac{64\,\frac{\sin(40^\circ)}{2}\cos(40^\circ)}{\cos(10^\circ)}$$
$$256\sin(10^\circ)\sin(30^\circ)\sin(50^\circ)\sin(70^\circ) = \frac{32\,\frac{\sin(80^\circ)}{2}}{\cos(10^\circ)}$$
$$256\sin(10^\circ)\sin(30^\circ)\sin(50^\circ)\sin(70^\circ) = \frac{16\cos(10^\circ)}{\cos(10^\circ)}$$
$$256\sin(10^\circ)\sin(30^\circ)\sin(50^\circ)\sin(70^\circ) = 16.$$

21

$\left(2+\sqrt{3}\right)^3 = \left(4+4\sqrt{3}+3\right)\left(2+\sqrt{3}\right) = \left(7+4\sqrt{3}\right)\left(2+\sqrt{3}\right) = 14+8\sqrt{3}+7\sqrt{3}+12 = 26+15\sqrt{3}$.

Knowing that $\sqrt{3} \approx 1,73$, $25 < 15\sqrt{3} < 26$ then $51 < \left(2+\sqrt{3}\right)^3 < 52$.

Otherwise, $\left(2+\sqrt{3}\right)^3 + \left(2-\sqrt{3}\right)^3 = \left(26+15\sqrt{3}\right) + \left(26-15\sqrt{3}\right) = 52$.

In fact $\left(2-\sqrt{3}\right)^3 = \left(4-4\sqrt{3}+3\right)\left(2-\sqrt{3}\right) = \left(7-4\sqrt{3}\right)\left(2-\sqrt{3}\right) = 14-8\sqrt{3}-7\sqrt{3}+12 = 26-15\sqrt{3}$.

Thus as $0 < 2-\sqrt{3} < 1$, $0 < \left(2-\sqrt{3}\right)^3 < 1$ so that we have $52-1 < \left(2+\sqrt{3}\right)^3 < 52$.

The greatest integer less than or equal to $\left(2+\sqrt{3}\right)^3$ is 51.

22

Method 1:

The numbers x_1, x_2 and x_3 are roots of $P(x) = (11-x)^3 + (13-x)^3 - (24-2x)^3 = 0$.
So we know that $P(x) = a(x-x_1)(x-x_2)(x-x_3)$ where a is the coefficient of the term in x^3.
Expanding $(11-x)^3 + (13-x)^3 - (24-2x)^3$, we have $P(x) = 6x^3 - 216x^2 + \ldots$. Then $a = 6$, hence $P(x) = 6(x-x_1)(x-x_2)(x-x_3) = 6x^3 - 6(x_1+x_2+x_3)x^2 + \ldots$ expanding the factored form.

So we deduce that $x_1 + x_2 + x_3 = \dfrac{216}{6} = 36$.

Method 2:

Let $a = 11-x$ and $b = 13-x$. The equality gives $a^3 + b^3 = (a+b)^3$.
So as $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, $ab(a+b) = 0$.
$a = 0$ gives $x = 11$, $b = 0$ gives $x = 12$ and finally $a+b = 0$ gives $x = 12$.
So the sum of the roots is $11 + 12 + 13 = 36$.

Remark:

The method 1 uses a general result on polynomials.
A n-degree polynomial in the real field is an expression of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ with $a_i \in \mathbb{R}$, $1 \le i \le n$ and $a_n \ne 0$.
A root of a real polynomial is a real value x_1 such as $P(x_1) = 0$.
If x_1 is a root of the polynomial $P(x)$ then $P(x) = (x-x_1)Q(x)$ where $Q(x)$ is a $(n-1)$-degree polynomial.

We demonstrate that a n-degree polynomial has at the most n roots.
Notice that all n-degree polynomial doesn't have n roots (for instance x^2+1 is a 2-degree polynomial and has no real roots).
In case that the n-degree polynomial $P(x)$ has n roots then $P(x) = a_n(x-x_1)(x-x_2)\ldots(x-x_n)$.
Developping this expression, we obtain:

$$P(x) = a_n x^n - a_n(\underbrace{x_1 + \ldots + x_n}_{\text{sum of the roots}})x^{n-1} + a_n(\underbrace{x_1x_2 + \ldots + x_{n-1}x_n}_{\text{sum of the products of the roots taken 2 by 2}})x^{n-2} + .$$

$$\ldots + a_n(\underbrace{x_1 \times \ldots \times x_{n-1} + \ldots + x_2 \times \ldots \times x_n}_{\text{sum of the products of the roots taken }(n-1)\text{ by }(n-1)})x + a_n x_1 \times \ldots \times x_n$$

We so have a direct relation between the roots and the coefficients of the polynomial.

For a quadratic (2-degree) polynomial with 2 roots:
$P(x) = a_2x^2 - a_2(x_1+x_2)x + a_2x_1x_2$.

For a cubic (3-degree) polynomial with 3 roots:
$P(x) = a_3x^3 - a_3(x_1+x_2+x_3)x^2 + a_3(x_1x_2+x_1x_3+x_2x_3)x + a_3x_1x_2x_3$.

For a 4-degree polynomial with 4 roots:

$P(x) = a_4 x^3 - a_4(x_1 + x_2 + x_3 + x_4) x^2 + a_4(x_1 x_2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4) x^2.$
$+a_3(x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4) x + a_3 x_1 x_2 x_3 x_4$

And so on…

23

We easily deduce that $\angle ACB = 70°$, $\angle MAC = 20°$ and $\angle MCB = 40°$.

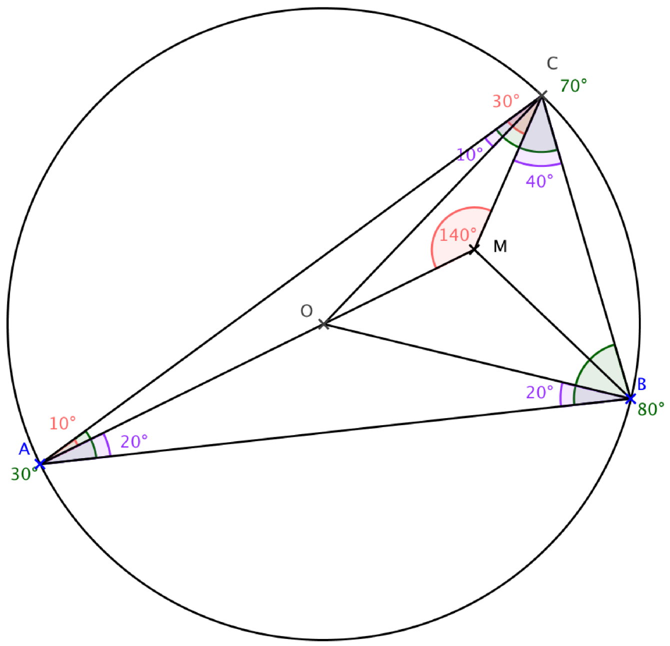
Lets introduce the point O center of the circumscribed circle of the triangle.
As O is the center of the circle and $\angle BAC$ and $\angle BOC$ intercepting the same chord BC, using the theorem of the inscried and central angles, $\angle BOC = 2\angle BAC = 2\times 30° = 60°$.
As $OB = OC$, the triangle OBC is isosceles and as $\angle BOC = 60°$, the triangle OBC is equilateral.
Thus $\angle OCB = \angle OBC = 60°$ and then $\angle OCA = 70° - 60° = 10°$ and $\angle OBA = 80° - 60° = 20°$.

As $\angle OCA = 10°$ and the triangle OAC is isosceles, $\angle CAO = 10°$.
As $\angle CAM = 10°$ and $\angle CAO = 10°$, the points A, O and M are on the same straight line.

As $\angle AOC = 160°$, then $\angle COM = 20°$.
As $\angle OCM = 30°$ and $\angle OCA = 10°$, $\angle MCO = 20°$.
The triangle MCO is isosceles in M.

So $MO = MC$ and $BO = BC$ so line BM is the perpendicular bissector of line segment OC.
As triangle OBC is equilateral, line BM is also the angle bissector of $\angle OBC$.
Therefore $\angle OBM = \angle MBC = 30°$.

Finally $\angle BMC = 180° - 40° - 30° = 110°$.



Note:
We easily deduce relations between the unknown angles $\angle AMB$, $\angle ABM$, $\angle MBC$ and $\angle BMC$.
But we always miss a relation to give a conclusion.
The relation is hereby obtained using the point O.
The angle $\angle BAC = 30°$ is determinant: it gives this special configuration of triangle OBC equilateral.

24

Note:
For all real x, $[x]$ is the integer part of x, it is define as the greatest integer less than or equal to x.

It is the unique integer such as $[x] \le x < [x] + 1$.

We know $\left[\frac{n}{2}\right] \le \frac{n}{2}$, $\left[\frac{n}{3}\right] \le \frac{n}{3}$ and $\left[\frac{n}{6}\right] \le \frac{n}{6}$.

So $\left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \left[\frac{n}{6}\right] \le n$ and as first $\left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \left[\frac{n}{6}\right] = n$ and second $\frac{n}{2} + \frac{n}{3} + \frac{n}{6} = n$, we can assure that $\left[\frac{n}{2}\right] = \frac{n}{2}$, $\left[\frac{n}{3}\right] = \frac{n}{3}$ and $\left[\frac{n}{6}\right] = \frac{n}{6}$.

In fact if any of $\left[\frac{n}{i}\right]$ is different from $\frac{n}{i}$, then it will imply that $\left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \left[\frac{n}{6}\right] < n$.

As $\left[\frac{n}{6}\right] \in \mathbb{N}$ and as $\left[\frac{n}{6}\right] = \frac{n}{6}$, 6 is a dividor of n (also 2 and 3, but $6 = 2 \times 3$, so 6 is the most restrictive condition).

Or $2007 = 334 \times 6 + 3$, so there are 334 multiple of 6 less than 2007.

There are 334 positive integer n less than 2007 such as $\left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \left[\frac{n}{6}\right] = n$.

25

The equation $\frac{b}{c-a} - \frac{a}{b+c} = 1$implies $c - a \ne 0$ and so is equivalent to $b(b+c) - a(c-a) = (c-a)(b+c)$.

Developping, we obtain $a^2 + b^2 - c^2 = -a\,b$.

Applying the law of cosines to $\Delta\, A\,B\,C$, we also have $c^2 = a^2 + b^2 - 2\,a\,b\cos(\hat{C})$ so $\cos(\hat{C}) = \frac{a^2 + b^2 - c^2}{2\,a\,b}$.

Thus combining the 2 results, $\cos(\hat{C}) = -\frac{1}{2}$.

Therefore $\hat{C} = 120\,°$. (it is necessarly the greatest angle).

So the value of the greatest angle of $\Delta\, A\,B\,C$ is 120º.

26

N is divisible by 10 or 12 is equivalent to N is a multiple of 10 or a multiple of 12.

$2007 = 200 \times 10 + 7$: thus there are 200 multiples of 10 between 1 and 2007.

$2007 = 167 \times 12 + 3$: thus there are 167 multiples of 12 between 1 and 2007.

Remains to count how many multiples of 10 are also multiples of 12.
As $10 = 2 \times 5$ and $12 = 2^2 \times 3$, a common multiple of 10 and 12 is a multiple of $\text{lcm}(10, 12) = 2^2 \times 3 \times 5 = 60$.
Given that $2007 = 33 \times 60 + 27$, there are 33 multiples of 60 between 1 and 2007.

Therefore the inclusion-exclusion principle gives that the number of multiples of 10 or 12 (or both) between 1 and 2007 is $200 + 167 - 33 = 334$.

27

We have $(x-a)(x-b) = x^2 + x\sin(\alpha) + 1$ and $(x-c)(x-d) = x^2 + x\cos(\alpha) - 1$.

Developping the factored expressions, we deduce: $\begin{cases} a\,b = 1 \\ a + b = -\sin(\alpha) \end{cases}$ and $\begin{cases} c\,d = -1 \\ c + d = -\cos(\alpha) \end{cases}$ (so a, b, c and d are all nonzero).

Though $a = \frac{1}{b}$ and $c = \frac{1}{d}$.

Thus $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}=b^2+a^2+d^2+c^2$.

Hence $(a+b)^2=a^2+2\,a\,b+b^2$,

$a^2+b^2=(-\sin(\alpha))^2-2\times 1=\sin^2(\alpha)-2$ and similarly $c^2+d^2=(-\cos(\alpha))^2-2\times(-1)=\cos^2(\alpha)+2$.

Finally $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}=\sin^2(\alpha)-2+\cos^2(\alpha)+2=\cos^2(\alpha)+\sin^2(\alpha)=1$.

28

Lets observe that the sequence (a_n) is periodic.

- $a_1=2$,
- $a_2=\frac{1+2}{1-2}=-3$,
- $a_3=\frac{1-3}{1+3}=-\frac{1}{2}$,
- $a_4=\frac{1-\frac{1}{2}}{1+\frac{1}{2}}=\frac{\frac{1}{2}}{\frac{3}{2}}=\frac{1}{3}$,
- $a_5=\frac{1+\frac{1}{3}}{1-\frac{1}{3}}=\frac{\frac{4}{3}}{\frac{2}{3}}=2=a_1$.

So we can deduce, by induction, that the sequence (a_n) is periodic with period 4.

Thus:

- if $n=4\,k$, $a_n=\frac{1}{3}$,
- if $n=4\,k+1$, $a_n=2$,
- if $n=4\,k+2$, $a_n=-3$,
- if $n=4\,k+3$, $a_n=-\frac{1}{2}$.

As $2007=501\times 4+3$, then $a_{2007}=-\frac{1}{2}$ and so $-2008\,a_{2007}=1004$.

29

We have:

$x\,y=\frac{(x+y)^2-(x^2+y^2)}{2}$ or $x\,y=\frac{(x^2+y^2)-(x-y)^2}{2}$.

$y\,z=\frac{(y+z)^2-(y^2+z^2)}{2}$ or $y\,z=\frac{(y^2+z^2)-(y-z)^2}{2}$.

$x\,z=\frac{(x+z)^2-(x^2+z^2)}{2}$ or $x\,z=\frac{(x^2+z^2)-(x-z)^2}{2}$.

So

$$4=x\,y+y\,z+x\,z=\frac{(x+y)^2-(x^2+y^2)}{2}+\frac{(y+z)^2-(y^2+z^2)}{2}+\frac{(x+z)^2-(x^2+z^2)}{2}=\text{ or}$$

$$\frac{(x+y)^2+(y+z)^2+(z+x)^2}{2}-\left(x^2+y^2+z^2\right)$$

$4 = xy + yz + xz = \left(x^2 + y^2 + z^2\right) - \dfrac{(x-y)^2 + (y-z)^2 + (z-x)^2}{2}$.

Thus:

$x^2 + y^2 + z^2 = \dfrac{(x+y)^2 + (y+z)^2 + (z+x)^2}{2} - 4 \geq -4$: this is not really convincing as $x^2 + y^2 + z^2 \geq 0$.

or

$x^2 + y^2 + z^2 = \dfrac{(x-y)^2 + (y-z)^2 + (z-x)^2}{2} + 4 \geq 4$: this is more interesting.

The least possible value is 4.

Remark:

We could have thought to use $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$ but it gives also an unsatisfactory result: $x^2 + y^2 + z^2 \geq -811$.

30

Note:

It is of common use that $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$.

Lets count the cases.
As $a_1 + 6 \leq a_2 + 3$, $a_1 + 3 \leq a_2$.
As $a_1 + 6 \leq a_3 \leq 15$ so $a_1 \leq 9$: $a_1 \in \{1; \ldots; 9\}$.
As $a_2 + 3 \leq a_3 \leq 15$ so $a_2 \leq 12$: $a_2 \in \{1; \ldots; 12\}$.

Suppose $a_1 = 1$, then $a_2 \geq 4$: $a_2 \in \{4; \ldots 12\}$.

- Suppose then $a_2 = 4$, as $a_2 + 3 \leq a_3$, $a_3 \geq 7$: $a_3 \in \{7; \ldots; 15\}$.
  All the triples $(1; 4; a_3\}$ with $a_3 \in \{7; \ldots; 15\}$ are solutions: there are 9 triples.
- Suppose then $a_2 = 5$, as $a_2 + 3 \leq a_3$, $a_3 \geq 8$: $a_3 \in \{8; \ldots; 15\}$.
  All the triples $(1; 5; a_3\}$ with $a_3 \in \{8; \ldots; 15\}$ are solutions: there are 8 triples.
- Suppose then $a_2 = 6$, as $a_2 + 3 \leq a_3$, $a_3 \geq 9$: $a_3 \in \{9; \ldots; 15\}$.
  All the triples $(1; 5; a_3\}$ with $a_3 \in \{9; \ldots; 15\}$ are solutions: there are 7 triples.

  … and so on until
- Suppose then $a_2 = 12$, as $a_2 + 3 \leq a_3$, $a_3 \geq 15$: $a_3 \in \{15; \ldots; 15\}$.
  All the triples $(1; 12; a_3\}$ with $a_3 \in \{15; \ldots; 15\}$ are solutions: there is 1 triple.

Finally there are $9 + 8 + 7 + \ldots + 1 = \dfrac{9 \times 10}{2} = 45$ triples solutions with $a_1 = 1$.

Suppose $a_2 = 2$, then $a_2 \geq 5$: $a_2 \in \{5; \ldots 12\}$.

- Suppose then $a_2 = 5$, as $a_2 + 3 \leq a_3$, $a_3 \geq 8$: $a_3 \in \{8; \ldots; 15\}$.
  All the triples $(2; 5; a_3\}$ with $a_3 \in \{8; \ldots; 15\}$ are solutions: there are 8 triples.

  … and so on until
- Suppose then $a_2 = 12$, as $a_2 + 3 \leq a_3$, $a_3 \geq 15$: $a_3 \in \{15; \ldots; 15\}$.
  All the triples $(2; 12; a_3\}$ with $a_3 \in \{15; \ldots; 15\}$ are solutions: there is 1 triple.

Finally there are $8 + 7 + \ldots + 1 = \dfrac{8 \times 9}{2} = 36$ triples solutions with $a_1 = 2$.

We proceed in the same way until $a_1 = 9$.
Then there is only one triple $(9; 12; 15)$ solution.

We finally get:

$\dfrac{9\times 10}{2} + \dfrac{8\times 9}{2} + \dfrac{7\times 8}{2} + \dfrac{6\times 7}{2} + \dfrac{5\times 6}{2} + \dfrac{4\times 5}{2} + \dfrac{3\times 4}{2} + \dfrac{2\times 3}{2} + \dfrac{1\times 2}{2}$ = triples solutions.

$45 + 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 = 165$

31

The binomial formula gives:
$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
$(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.
So:
$(x+y)^4 + (x-y)^4 = 2x^4 + 12x^2y^2 + 2y^4$.
As we know that $(x+y)^4 + (x-y)^4 = 4112$, we deduce that $2x^4 + 12x^2y^2 + 2y^4 = 4112$ so $x^4 + 6x^2y^2 + y^4 = 2056$.

Then $(x^2-y^2)^2 = x^4 - 2x^2y^2 + y^4$ so as $x^2 - y^2 = 16$, we obtain $x^4 - 2x^2y^2 + y^4 = 16^2 = 256$.
Substituing $x^4 + y^4$ by $256 + 2x^2y^2$, we obtain $8x^2y^2 + 256 = 2056$ so $x^2y^2 = 225$.

Yet, $(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2 = 16^2 + 4\times 225 = 1156$.

Then $x^2 + y^2 = \sqrt{1156} = 34$.

32

As $\tan(8A) = \dfrac{\sin(8A)}{\cos(8A)}$, $\tan(8A) = \dfrac{\cos(A) - \sin(A)}{\cos(A) + \sin(A)}$ gives $\sin(8A)(\cos(A) + \sin(A)) = \cos(8A)(\cos(A) - \sin(A))$.
So $\sin(8A)\cos(A) + \cos(8A)\sin(A) = \cos(8A)\cos(A) - \sin(8A)\sin(A)$.
Hence $\sin(9A) = \cos(9A)$ and finally $\tan(9A) = 1$.

Yet $\tan(\alpha) = 1$ if $\alpha = 45°$ (modulo $180°$).
As a result, assuming $A > 0$, $9A = 45°$ and then $x = 5$ is the smallest value possible.

33

We know that $\left|n-k\right| = \begin{cases} n-k & \text{if } n \geq k \\ -(n-k) = k-n & \text{if } n < k \end{cases}$.

First case: $n \geq 100$.

$$\sum_{k=1}^{100}\left|n-k\right| = \sum_{k=1}^{100}(n-k) = (n-1) + (n-2) + \ldots + (n-100) = 100n - \frac{100\times 101}{2} = 100n - 5050.$$

The minimum is clearly occuring for $n = 100$ and worth 4950.

Second case: $n < 100$.

$$\sum_{k=1}^{100}\left|n-k\right| = \sum_{k=1}^{n}\left|n-k\right| + \sum_{k=n+1}^{100}\left|n-k\right| = \sum_{k=1}^{n}(n-k) + \sum_{k=n+1}^{100}(k-n) =.$$

$$(n-1) + \ldots + (n-n) + (n+1-n) + \ldots + (100-n)$$

$$\sum_{k=1}^{100}|n-k| = (n-1) + \underbrace{\ldots + 0) + (n+1) + }_{\text{100 premiers entiers sauf } n}\ldots + 100 - n\times(100-n) = \frac{100\times 101}{2} - n + n^2 - 100n = n^2 - 101n + 5050\ .$$

We then know that the minimum of a 2-degree polynomial function $f(x) = ax^2 + bx + c$, $a > 0$, is reached for $x = -\dfrac{b}{2a}$.

So the minimum of the function $f(x) = x^2 - 101x + 5050$ is reached at $x = -\dfrac{-101}{2\times 1} = 50,5$.

As n is a integer, it falls $n = 50$ or $n = 51$.

For $n = 50$, $50^2 - 101\times 50 + 5050 = 2500$ and for $n = 51$, $51^2 - 101\times 51 + 5050 = 2500$ (by symetry).

Finally the minimum of $\sum_{k=1}^{100} | n - k |$ is 2500, reached for $n = 50$ either $n = 51$.

34

As $2x + 3y = 2007$, then $y = \frac{2007 - 2x}{3} = 900 - \frac{2x}{3}$ and $2x = 3\times(669 - y)$.

As x and y are integers and as 2 and 3 are prime numbers, it implies that x is a multiple of 3: $x = 3k$ with k positive integer.

We write $2x = 3\times(669 - y)$, $2\times 3t = 3\times(669 - y)$ and we have $y = 669 - 2t$.

As $y \geq 0$, then $669 - 2t > 0$ so $t < 334,5$ consequently $t \leq 334$ (t integer).

Reciprocally, let consider the pair $(x; y) = (3t;\ 669 - 2t)$ for $0 < t \leq 334$.

Yet $2x + 3y = 2\times 3t + 3\times 669 - 3\times 2t = 2007$.

So such pair is a solution to the equation.

In conclusion, we have 334 solutions.

35

Hocus-pocus:

As we already seen before, $\frac{1}{k} - \frac{1}{k+1} = \frac{1}{k(k+1)}$, we have $\frac{1}{1 + k + k^2} = \frac{1}{2}\left(\frac{1}{(k-1)k + 1} - \frac{1}{k(k+1)+1}\right)$.

Thinking to such trick, isn't always easy.

$$S = \frac{1}{2}\left(\left(\frac{1}{0\times 1 + 1} - \frac{1}{1\times 2 + 1}\right) + \left(\frac{1}{1\times 2 + 1} - \frac{1}{2\times 3 + 1}\right) + \ldots + \left(\frac{1}{199\times 200 + 1} - \frac{1}{200\times 201 + 1}\right)\right) =.$$

$$\frac{1}{2}\left(1 - \frac{1}{200\times 201 + 1}\right)$$

$$S = \frac{1}{2}\times\frac{(40\,201 - 1)}{40\,201} = \frac{20\,100}{40\,201}.$$

Finally $80\,402\,S = 2\times 40\,201\times\frac{20\,100}{40\,201} = 40\,200$.