Maths' Lab: SMO 2010 Open

Maths' Lab: Elements of solutions.

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1 Answer: 50.

We have $8n^3 - 96n^2 + 360n - 400 = (2n-7)(4n^2 - 34n + 61) + 27$ so $\dfrac{8n^3 - 96n^2 + 360n - 400}{2n-7} = 4n^2 - 34n + 61 + \dfrac{27}{2n-7}$.

So as for all n, $4n^2 - 34n + 61 \in \mathbb{Z}$, then $\dfrac{8n^3 - 96n^2 + 360n - 400}{2n-7}$ is an integer if and only if $\dfrac{27}{2n-7}$ is also an integer.

Thus, if $|2n-7| > 27$, it is $n > 17$ or $n < -10$, $0 < \left|\dfrac{27}{2n-7}\right| < 1$ so no integer $n > 17$ fits.

So $n \in [\![-10; 17]\!]$ such that $\dfrac{27}{2n-7} \in \mathbb{Z}$ so such that $2n-7$ divide 27.

As $27 = 3^3$, divisors of 27 are $-27; -9; -3; -1; 1; 3; 9; 27$ so n is such that $2n-7 \in \{-27; -9; -3; -1; 1; 3; 9; 27\}$.
It comes $n \in \{-10; -1; 2; 3; 4; 5; 8; 17\}$.

So $\sum_{n\in S} |n| = 10 + 1 + 2 + 3 + 4 + 5 + 8 + 17 = 50$.

Remark:
In terms of congruences, we must have $2n - 7 \equiv 0 \ (27)$ so $2n \equiv 7 \ (27)$, $2n \equiv -20 \ (27)$ and as 2 and 27 are relatively primes, $n \equiv -10 \ (27)$, $n \equiv 17 \ (27)$.
Then $n = 27k + 17$ such that $-27 \le 27k + 17 \le 27$
So $-44 \le 27k \le 10$.

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2 Answer: 199.

$|x^2 - 4x - 39\,601| \ge |x^2 + 4x - 39\,601| \Longleftrightarrow (x^2 - 4x - 39\,601)^2 - (x^2 + 4x - 39\,601)^2 \ge 0$.

As $(x^2 - 4x - 39\,601)^2 - (x^2 + 4x - 39\,601)^2 = -16x(x^2 - 199^2) = -16x(x+199)(x-199)$, we obtain $|x^2 - 4x - 39\,601| \ge |x^2 + 4x - 39\,601| \Longleftrightarrow x(x+199)(x-199) \le 0$.

It is then trivial that 199 is the largest value of x such that $x(x+199)(x-199) \le 0$.

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3 Answer: 1159.

As $20^3 = 8000$, for all integer $k \in \{1; \ldots; 7999\}$, it exists $a \in \{1; \ldots; 19\}$ such that $a^3 \le k < (a+1)^3$.
Thus for all integer $k \in \{1; \ldots; 7999\}$, it exists $a \in \{1; \ldots; 19\}$ such that $a \le k^{1/3} < (a+1)$ and so $\lfloor k^{1/3} \rfloor = a$.

Therefore $x = \sum_{1^3 \le k < 2^3} 1 + \sum_{2^3 \le k < 3^3} 2 + \ldots + \sum_{19^3 \le k < 20^3} 19$.

Let's remark that there are $(a+1)^3 - a^3$ numbers between a^3 and $(a+1)^3$.

S o

$x = 1\times(2^3 - 1^3) + 2\times(3^3 - 2^3) + \ldots + 18\times(19^3 - 18^3) + 19\times(20^3 - 19^3)$.

$x = -1^3 + 2^3 - 2\times2^3 + 2\times3^3 - 3\times3^3 + \ldots + 18\times19^3 - 19\times19^3 + 19\times20^3$

$x = 19 \times 20^3 - \sum_{k=1}^{19} k^3$.

Let's remind that $\sum_{k=1}^{n} k^3 = \frac{1}{4} n^2(n+1)^2$ so $x = 19 \times 8000 - \frac{1}{4} \times 19^2 \times 20^2$.

Finally $x = 115\,900$ and $\left\lfloor \frac{x}{100} \right\rfloor = 1159$.

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4 Answer: 65.

For all integer $n \geq 6$, $\frac{6^n}{n!} = \frac{6}{1} \times \frac{6}{2} \times \frac{6}{3} \times \frac{6}{4} \times \frac{6}{5} \times \frac{6}{6} \times \frac{6}{7} \times \ldots \times \frac{6}{k} \times \ldots \times \frac{6}{n} \leq \frac{6}{1} \times \frac{6}{2} \times \frac{6}{3} \times \frac{6}{4} \times \frac{6}{5}$ as $0 < \frac{6}{k} < 1$ for all integer $6 < k \leq n$.

We have $\frac{6^1}{1!} = 6$, $\frac{6^2}{2!} = 18$, $\frac{6^3}{3!} = 36$, $\frac{6^4}{4!} = 54$ and $\frac{6^5}{5!} = 64$, $8 < 65$.

So the smallest positive integer C such that $\frac{6^n}{n!} \leq C$ for all positive integers n is 65.

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5 Answer: 78.

Theorem: Power of a point with respect to a circle.
With Γ a circle and P a point of the plane and d a line through P intersecting the circle Γ at points A and B, the number $PA \times PB$ is independant from line d equal to $s^2 - r^2$ where s is the distance from P to the centre of the circle Γ and r the radius of Γ.
If d is tangent to circle Γ at T then $PT^2 = s^2 - r^2$.
Let's note that if P is exterior to Γ, this number is positive, null if P is on Γ and negative if P is interior to Γ.
This number is called the power of point P with respect to circle Γ.

Let H be the point of Γ_2 such that segment HM is a diameter of circle Γ_2.
Thus the power of point M with respect to circle Γ_2 is given by $MH \times MN = MP \times MQ$.
The power of M with respect to circle Γ_1 is given by $MC \times MD = MP \times MQ : MH \times MN = MC \times MD$.
(In fact as M is on line PQ and as P and Q are belonging to both circles Γ_1 and Γ_2, the power of point M with respect to circle Γ_1 is equal to the power of P with respect to circle Γ_2.)
Let's note that $MD = MN + 60$ and $MH = MC + 60$ so we have $(MC + 60) \times MN = MC \times (MN + 60)$, $60\,MN = 60\,MC$ and finally $MN = MC$.
As $MN + MC = NC = 60$, it falls $MN = MC = 30$. M is the midpoint of segment CN.

The power of N with respect to circle Γ_1 is given by $NC \times ND = NA \times NB$.
So $60^2 = NA \times (2 \times 61 - NA)$.
NA is solution of the quadratic equation $x^2 - 122x + 60^2 = 0$.
The discriminant of this equation is
$\Delta = 122^2 - 4 \times 60^2 = (2 \times (60+1))^2 - 4 \times 60^2 = 4 \times 60^2 + 8 \times 60 + 1 - 4 \times 60^2 = 484 = 22^2$.
So $NA = \frac{122 - 22}{2} = 50$ or $NA = \frac{122 + 22}{2} = 72$.
As $NA > NB$, and $122 = AB = NA + NB$, then $NA = 72$.

Using Pythagoreas Theorem in the triangle AMN right-angled at N, $AM^2 = 30^2 + 72^2 = 6084 = 78^2$.

Remark:
Finding the saqure root of a number, manually isn't so difficult.
In fact for instance, as $6084 < 10\,000$, $\sqrt{6084} < 100$.
Supposing it is an integer, we cant write it $\sqrt{6084} = 10a + b$, with $a, b \in \{0; 1; \ldots; 9\}$.

So $(10a + b)^2 = 100a^2 + 20ab + b^2 = 6084$.
As $7^2 = 49 < 60$ and $8^2 = 64 > 60$, $a = 7$ so $10a + b = 70 + b$.
So $(70 + b)^2 = 4900 + 140b + b^2 = 6084 = 4900 + 1184$.
As $b^2 \leq 81$, we need $140b \geq 1000$ so $b = 8$ or $b = 9$ (in fact $140\times 7 = 980$).
Therefore as the last digit is 4, $b = 8$.
We have $140\times 8 + 8^2 = 1120 + 64 = 1184$.

Another way is to decompose 6084 in prime factor.
$6084 = 2^2\times 3^2\times 13^2$ so $\sqrt{6084} = 2\times 3\times 13 = 78$.

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6 Answer: 2500.

Theorem: inequality of Cauchy-Schwarz.
For all real numbers $(a_k)_{1\leq k\leq n}$ et $(b_k)_{1\leq k\leq n}$, $\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$ where equality holds for $b_k = c\,a_k$ for all k .

As $x_k \geq 0$ for $k = 1; \ldots; 50$, with $a_k = \sqrt{x_k}$ and $b_k = \frac{1}{\sqrt{x_k}}$, the Cauchy-Schwarz inequality yields to

$$\left(\sum_{k=1}^{50}\sqrt{x_k}\times\frac{1}{\sqrt{x_k}}\right)^2 \leq \sum_{k=1}^{50}\left(\sqrt{x_k}\right)^2\sum_{k=1}^{50}\left(\frac{1}{\sqrt{x_k}}\right)^2 \text{ so } \left(\sum_{k=1}^{50}1\right)^2 \leq \sum_{k=1}^{50}x_k\times\sum_{k=1}^{50}\frac{1}{x_k} \text{ and finally } 50^2 \leq \sum_{k=1}^{50}x_k .$$

As the equality holds for $x_k = 50$ for all k ($\sum_{k=1}^{50}50 = 50\times 50$),then the minimum value of $\sum_{k=1}^{50}x_k$ is $50^2 = 2500$.

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7 Answer: 26.

Supposing $(x - p)(x - 13) + 4 = (x + q)(x + r)$ yields for $x = -q$ to $(-q - p)(-q - 13) = -4$ so $(q + p)(q + 13) = -4$.
Consequently as p and q are integers, we must have the following cases:

- $q + p = 4$ and $q + 13 = -1$ so $q = -14$ and $p = 18$
- $q + p = -4$ and $q + 13 = 1$ so $q = -12$ and $p = 8$
- $q + p = 2$ and $q + 13 = -2$ so $q = -15$ and $p = 17$
- $q + p = -2$ and $q + 13 = 2$ so $q = -11$ and $p = 9$.

As $(x - p)(x - 13) + 4 = (x + q)(x + r)$, then $13p + 4 = qr$.
Therefore $r = \frac{13p + 4}{q}$ and we must have $r \neq q$ and r integer.
The first case gives $r = \frac{13\times 18 + 4}{-14} = -17$, the second $r = \frac{13\times 8 + 4}{-12} = -9$, the third $r = \frac{13\times 17 + 4}{-15} = -15 = q$ and the fourth $r = \frac{13\times 9 + 4}{-11} = -11 = q$.
Therefore there are only two solutions $p = 18$ or $p = 8$. The sum is $8 + 18 = 26$.

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8 Answer: 8038.

For all integer k, $p_k = 1 - \frac{1}{k^2} + \frac{1}{k}\left(1 - \frac{1}{k^2}\right) = \left(1 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{k}\right) = \left(1 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right)^2 = \frac{(k-1)(k+1)^2}{k^3}$.

It yields to $p_2 \dots p_n = \frac{1\times 3^2}{2^3} \times \frac{2\times 4^2}{3^3} \times \frac{3\times 5^2}{4^3} \times \dots \times \frac{(n-2)\,n^2}{(n-1)^3} \times \frac{(n-1)\,(n+1)^2}{n^3} = \frac{(n+1)^2}{4\,n}$.

We must find n such that $\frac{(n+1)^2}{4\,n} > 2010$ so, as n is a strictly positive integer, we must have n such that $n^2 + 2\,n + 1 - 2010\times 4\,n > 0$ and so $n^2 - 8038\,n + 1 > 0$.
Then $n(n-8038) > -1$ and as n is a strictly positive integer, $n(n-8038) \geq 0$ so $n \geq 8038$.

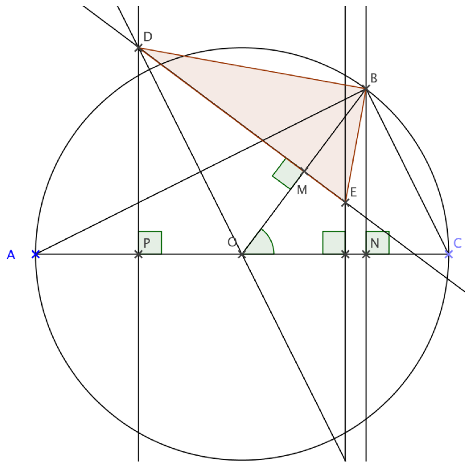
9 Answer: 45.

Let M be the foot of the perpendicular bissector of segment BO.

Thus M is the midpoint of BO so $BM = \frac{1}{2}\,OB = \frac{1}{4}\,AC = 6$.

As D and E are the circumcentres of the triangles OAB and OBC then M, D, E are aligned.
M is therefore the foot of the height issued by B in triangle BDE .

So $A_{BDE} = \frac{1}{2}\,DE\times BM = 3\,DE$.



As D is the circumcentre of triangle OAB, the inscribded angle theorem shows that $\angle ODB = 2\angle CAB$ and as triangle ODB is isosceles, $\angle EDB = \frac{1}{2}\angle ODB = \angle CAB$.

Identically, we obtain $\angle DEB = \angle ACB$.
Finally triangles BDE and ABC are similar. In fact triangle BAC is right-angled so $\angle EDB + \angle DEB = \angle CAB + \angle ACB = \frac{\pi}{2}$ and then $\angle DBE = \frac{\pi}{2}$: triangle DBE is right-angled.
The two triangles have the same angles. They are similar.

We deduce $\frac{DE}{AC} = \frac{BM}{BN}$ with N the foot of the height relative to edge B in triangle ABC.

Yet $BN = OB\sin(\angle COB) = 12\times\frac{4}{5} = \frac{48}{5}$.

Hence $DE = 24\times\frac{6}{\frac{48}{5}} = 15$.

We deduce $A_{BDE} = 3\times 15 = 45$.

10 Answer: 4.

Note that $(x+1)^3 + 3\,(x+1) = x^3 + 3\,x^2 + 6\,x + 4$ so $f(x) = (x+1)^3 + 3\,(x+1) + 10$.

We deduce $(a+1)^3 + 3\,(a+1) = -9$ and $(b+1)^3 + 3\,(b+1) = 9$.

The function $g: x \mapsto x^3 + 3x$ is an strictly increasing odd function.
As $g(a+1) = g(b+1)$ then $a+1 = -(b+1)$.
So $a + b = -2$: $(a+b)^2 = -4$.

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11 Answer: 11.

Let remain that $\tan(\alpha+\beta) = \dfrac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$ so

$$\tan(\alpha+\beta+\gamma) = \frac{\tan(\alpha)+\tan(\beta+\gamma)}{1-\tan(\alpha)\tan(\beta+\gamma)} = \frac{\tan(\alpha)+\frac{\tan(\beta)+\tan(\gamma)}{1-\tan(\beta)\tan(\gamma)}}{1-\tan(\alpha)\frac{\tan(\beta)+\tan(\gamma)}{1-\tan(\beta)\tan(\gamma)}} = \frac{\tan(\alpha)+\tan(\beta)+\tan(\gamma)-\tan(\alpha)\tan(\beta)\tan(\gamma)}{1-\tan(\beta)\tan(\gamma)-\tan(\alpha)\tan(\beta)-\tan(\gamma)\tan(\beta)}.$$

With $x = \tan(\alpha)$, $y = \tan(\beta)$ and $z = \tan(\gamma)$: $\tan(\alpha+\beta+\gamma) = \dfrac{x+y+z-xyz}{1-(xy+xz+yz)}$.

As $\cot = \dfrac{1}{\tan}$, we also have:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = -\frac{4}{5} \text{ so } \frac{xy+xz+yz}{xyz} = -\frac{4}{5}$$

$$x+y+z = \frac{17}{6}$$

$$\frac{1}{xy}+\frac{1}{yz}+\frac{1}{xz} = -\frac{17}{5} \text{ so } \frac{x+y+z}{xyz} = -\frac{17}{5} .$$

Thus we deduce $xyz = -\dfrac{5}{6}$ and $xy+xz+yz = \dfrac{2}{3}$ and finally $\tan(\alpha+\beta+\gamma) = \dfrac{\frac{17}{6}-\left(-\frac{5}{6}\right)}{1-\frac{2}{3}} = 11$.

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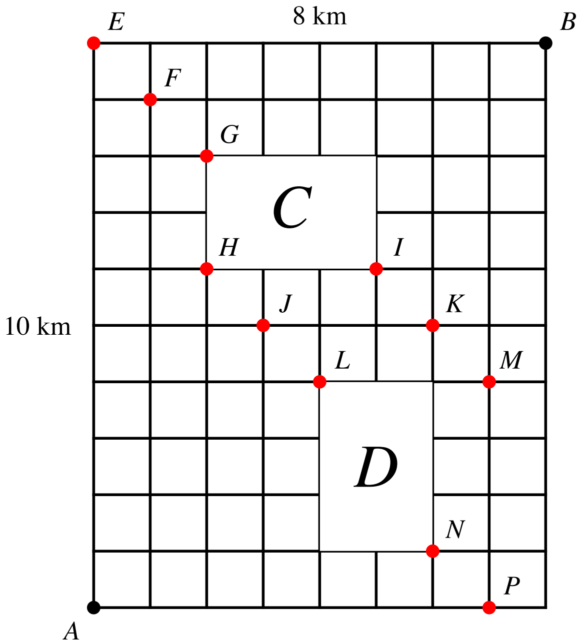
12 Answer: 22023.

Let note that the shortest routes are composed by 8 moves of 1 km to the left and 10 moves of 1 km to the top.
So the goal here is to count all the possible routes avoiding the two estates.
We consider the points as shown on the figure.



We have the routes:

- $A \to E \to B$: 1 route.
- $A \to F \to B$: $\binom{10}{1} \times \binom{8}{1} = 80$ routes.
- $A \to G \to B$: $\binom{10}{2} \times \binom{8}{2} = 1260$ routes.
- $A \to H \to I \to B$: $\binom{8}{2} \times \binom{7}{3} = 980$ routes.
- $A \to J \to I \to B$: $\binom{8}{3} \times \binom{3}{1} \times \binom{7}{3} = 5880$ routes.
- $A \to J \to K \to B$: $\binom{8}{3} \times \binom{7}{2} = 1176$ routes.
- $A \to L \to K \to B$: $\binom{8}{4} \times \binom{3}{1} \times \binom{7}{2} = 4410$ routes.
- $A \to L \to I \to B$: $\binom{8}{4} \times \binom{3}{1} \times \binom{7}{3} = 7350$ routes.
- $A \to L \to M \to B$: $\binom{8}{4} \times \binom{7}{1} = 490$ routes.
- $A \to N \to B$: $\binom{7}{1} \times \binom{11}{2} = 385$ routes.
- $A \to P \to B$: $\binom{11}{1} = 11$ routes.

We then have 1 + 80 + 1260 + 980 + 5880 + 1176 + 4410 + 7350 + 490 + 385 + 11 = 22 023 routes.

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13 Answer: 4021.

We have:

$$a_{n+1} - a_n = \frac{2n}{n+1} a_n - \frac{n-1}{n+1} a_{n-1} - a_n = \frac{1}{n+1}(2n a_n - (n-1) a_{n-1} - (n+1) a_n) = \frac{n-1}{n+1}(a_n - a_{n-1}).$$

So by induction, we obtain

$$a_{n+1} - a_n = \frac{n-1}{n+1} \times \frac{n-2}{n} \times \frac{n-3}{n-1} \times \ldots \times \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3}(a_2 - a_1) = \frac{2}{(n+1)n}(a_2 - a_1) = \frac{2}{(n+1)n}.$$

As $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, $a_{n+1} - a_n = \frac{2}{n} - \frac{2}{n+1}$.

Hence

$$a_n = \frac{2}{n-1} - \frac{2}{n} + a_{n-1}$$

$$a_n = \frac{2}{n-1} - \frac{2}{n} + \frac{2}{n-2} - \frac{2}{n-1} + a_{n-2}$$

$$a_n = \frac{2}{n-2} - \frac{2}{n} + \frac{2}{n-3} - \frac{2}{n-2} + a_{n-3}$$

$$\ldots$$

$$a_n = \frac{2}{3} - \frac{2}{n} + \frac{2}{2} - \frac{2}{3} + a_2 = 3 - \frac{2}{n}$$

$$a_n = 3 - \frac{2}{n}.$$

So the least value of m such that $a_n > 2 + \frac{2009}{2010}$ for all $n \geq m$ is the least value of m such that $3 - \frac{2}{m} > 2 + \frac{2009}{2010}$.

Then $\dfrac{2}{m} < \dfrac{1}{2010}$ and $m > 4020$. The least value is 4021.

14 Answer: 14.

15 Answer: 3.

Let f be the function define by $f(x) = x^5 - x^3 + x - 2$.

We have $f'(x) = 5x^4 - 3x^2 + 1$.

As the discriminant of $5X^2 - 3X + 1$ is $\Delta = -11 < 0$, $5X^2 - 3X + 1 > 0$ for all real X and for $X = x^2$, $5x^4 - 3x^2 + 1 > 0$ for all real x.
Thus the function f is strictly increasing. The polynomial equation $f(x) = 0$, because the degree of $f(x)$ is 5, has a unique solution over $\mathbb{R}$.
We have $f(1) = -1 < 0$ and $f(2) = 24 > 0$ so $1 < \alpha < 2$.

Then as $\alpha^5 = \alpha^3 - \alpha + 2$ and $\alpha^6 = \alpha^4 - \alpha^2 + 2\alpha$.
With $\lfloor \alpha^6 \rfloor = n$, $n \le \alpha^4 - \alpha^2 + 2\alpha < n + 1$ so $n\alpha \le \alpha^5 - \alpha^3 + 2\alpha^2 < (n+1)\alpha$ and $\alpha^5 = \alpha^3 - \alpha + 2$, it yields to $n\alpha \le 2\alpha^2 - \alpha + 2 < (n+1)\alpha$.
So n must be sucht that $P_1(\alpha) = 2\alpha^2 + (-1-n)\alpha + 2 \ge 0$ and such that $P_2(\alpha) = 2\alpha^2 + (-1-(n+1))\alpha + 2 < 0$.
The discrimants of the quadratics functions are $\Delta_1 = (-1-n)^2 - 4\times 2\times 2 = n^2 + 2n - 15$ and $\Delta_2 = n^2 + 4n - 12$.
For $n = 1$, $\Delta_1 = -12 < 0$ et $\Delta_2 = -7 < 0$: so $n = 1$ doesn't fit as $P_2(x) > 0$ for all real x .
For $n = 2$, $\Delta_1 = -7 < 0$ et $\Delta_2 = 0$: so $n = 2$ doesn't fit as $P_2(x) \ge 0$ for all real x .

For $n = 3$, $\Delta_1 = 0$ et $\Delta_2 = 9 > 0$. So $P_1(x) \ge 0$ for all real x and P_2 has to roots $x_1 = \dfrac{1}{2} < 1$ et $x_2 = 2$ so $P_2(\alpha) < 0$.

$n = 3$ can be the value.

For $n = 4$, $\Delta_1 = 9$ and $\Delta_2 = 20$. The roots of P_1 are $x_1 = \dfrac{1}{2} < 1$ and $x_2 = 2$ so $P_1(\alpha) < 0$: $n = 4$ doesn't fit.

We then deduce that $\lfloor \alpha^6 \rfloor = n = 3$.

16 Answer: 8030.

First notice that for all $k \ge 1$, $2k + 1 = (k+1)^2 - k^2$: thus all odd integer can be written as the difference of two squares.
No cute number else than 1 is odd.
Observing the even numbers: 2 is cute, 4 is cute, 6 is cute, $8 = 3^2 - 1^2$ isn't, 10 is cute $12 = 4^2 - 2^2$, 14 is cute, $16 = 5^2 - 9^2$,….
So it looks like that for $k > 1$, $4k + 2$ is cute but $4k$ isn't.
We notice that $4k = (k+1)^2 - (k-1)^2$ for all $k > 1$ and then $4k$ can be written as the difference of two squares.

Let k be a positive integer > 1.
Suppose that $4k + 2 = a^2 - b^2 = (a+b)(a-b)$ with a and b integers.
As $4k + 2$ is even, then $a + b$ and $a - b$ must be even too. In fact $a + b$ and $a - b$ have the same parity and the product of two odd numbers are odd.
It yields to $4k + 2 = 2p\times 2q = 4(pq)$: $4k + 2$ is divisible by 4: absurd.
Therefore all numbers of the form $4k + 2$, $k > 1$ are cute.

As all integers are or odd or of the form $4k$ or of the form $4k + 2$, thus the cute numbers are 1, 2, 4 and all the numbers of the form $4k + 2$, $k > 1$.

The 2010$^{\text{th}}$ cute number is then $4\times 2007 + 2 = 8030$.

17 Answer: 841.

We have:

- $f(x) = p_1(x)\,(x-1) + 3$ so $f(1) = 3$.
- $f(x) = p_2(x)\,(x-2) + 1$ so $f(2) = 1$.
- $f(x) = p_3(x)\,(x-3) + 7$ so $f(3) = 7$.
- $f(x) = p_4(x)\,(x-4) + 36$ so $f(4) = 36$.
- $f(x) = q(x)\,(x^2 - x - 1) + (x-1)$ with $q(x)$ a polynomial of degree 3.

As we can notice on the 4 first cases, $f(a)$ is the remainder of $f(x)$ divided by $(x-a)$.
Consequently the remainder of $f(x)$ divided by $(x+1)$ is equal to $f(-1)$.
Let's determine $f(-1)$.

The polynomial q is of degree 3 and we have 4 values.
The 4 first divisions yield to $-q(1) = 3$ so $q(1) = -3$, $q(2) = 0$, $q(3) = 1$ and $q(4) = 3$.

So with $q(x) = a\,x^3 + b\,x^2 + c\,x + d$, numbers a, b, c and d are solutions of the system: $\begin{cases} a+b+c+d=-3 \\ 8\,a+4\,b+2\,c+d=0 \\ 27\,a+9\,b+3\,c+d=1 \\ 64\,a+16\,b+4\,c+d=3 \end{cases}$.

We obtain $\begin{cases} a+b+c+d=-3 \\ 4\,b+6\,c+7\,d=-24 \\ 18\,b+24\,c+26\,d=-82 \\ 48\,b+60\,c+63\,d=-195 \end{cases}$ then $\begin{cases} a+b+c+d=-3 \\ 4\,b+6\,c+7\,d=-24 \\ 6\,c+11\,d=-52 \\ 12\,c+21\,d=-93 \end{cases}$ and finally $\begin{cases} a+b+c+d=-3 \\ 4\,b+6\,c+7\,d=-24 \\ 6\,c+11\,d=-52 \\ d=-11 \end{cases}$.

Thus $d=-11$, $c = \dfrac{-52+121}{6} = \dfrac{23}{2}$, $b = \dfrac{-24+7\times 11-6\times\frac{23}{2}}{4} = -4$ and $a = -3+4-\dfrac{23}{2}+11 = \dfrac{1}{2}$.

We have $q(x) = \dfrac{1}{2}\,x^3 - 4\,x^2 + \dfrac{23}{2}\,x - 11$.

So $f(x) = \left(\dfrac{1}{2}\,x^3 - 4\,x^2 + \dfrac{23}{2}\,x - 11\right)\left(x^2 - x - 1\right) + (x-1)$ and

$f(-1) = \left(-\dfrac{1}{2} - 4 - \dfrac{23}{2} - 11\right)\times 1 - 2 = -27 - 2 = -29$.

The square remainder of $f(x)$ divided by $(x+1)$ is $(-29)^2 = 841$.

18 Answer: 18.

We have $b = \dfrac{100\,a + 100}{a - 100}$.
As b is a positive integer, $a > 100$.
Let $a = 100 + a'$, where a' is a positive integer .
It yields to $b = \dfrac{100\times 100 + 100\,a' + 100}{a'} = 100 + \dfrac{10\,100}{a'}$.
So a' is a divisor of $10\,100 = 2^2\times 5^2\times 101$. There are $(2+1)\,(2+1)\,(1+1) = 18$ divisors of 10 100 .
Let notice that each couple $\left(100 + a',\ 100 + \frac{10\,100}{a'}\right)$ where a' is a divisor of 10 100 yields to a solution .
In fact $100\left(100 + a' + 100 + \dfrac{10\,100}{a'}\right) = 20\,000 + 100\,a' + \dfrac{1\,010\,000}{a'}$ and

$a\,b - 100 =$
$(100 + a')\left(100 + \dfrac{10\,100}{a'}\right) - 100 = 10\,000 + \dfrac{1\,010\,000}{a'} + 100\,a' + 10\,100 - 100 = 20\,000 + 100\,a' + \dfrac{1\,010\,000}{a'}$.

Consequently there are 18 couples satisfying the equation.

19 Answer: 17 .

Notice that the power of an odd number is still odd and that the sum of two odd numbers is even, then or p is even or p is odd and for instance a is even and b odd.
But p can not be even. In fact it will yield to $a = b = 2$ and $p > 2$ and p is divisible by 2.
Hence p is odd. As a and b are prime, and a is even $a = 2$.
We have $p = 2^b + b^2$.
By trial and error, with $b = 3$, b is prime and $2^3 + 3^2 = 17$ which is prime.
So $(a, b, p) = (2, 3, 17)$ is suitable.

Let show that 3 is the only possible value.
Let $b = 2k + 1$ with $k \geq 1$.
Suppose that $k \geq 2$.
If $k = 3k + 1$ then $b > 3$ and $b = 2k + 1 = 6k + 3$ is divisible by 3. Absurd because b is prime.
Suppose $k \not\equiv 1$ (modulo 3).
$p = 2^b + b^2 = 2^{2k+1} + (2k + 1)^2 = 2\times 4^k + 4k^2 + 2k + 1$.
As $4 \equiv 1$ (modulo 3), $2^b + b^2 \equiv 4k(k + 1) + 3 \equiv 4k(k + 1)$ (modulo 3) .
Yet or $k \equiv 0$ (modulo 3) then $2^b + b^2 \equiv 0$ (modulo 3): absurd as $p = 2^b + b^2$ is prime, or $k \equiv 2$ (modulo 3) then $(k + 1) \equiv 3$ (modulo 3) and again $p \equiv 0$ (modulo 3).
Absurd.
Therefore $k < 2$ and $k = 1$.

Reminder:
We note $a \equiv b$ (modulo n) the relation define by a and b have the same remainder in the euclidian division by n .

20 Answer: 10045.

Let consider the linear function $f(x) = \dfrac{11}{2010}x$.
For all integer $x \in \{1; 2; \ldots; 2009\}$, $f(x) \notin \mathbb{N}$.

Thus for all integer $x \in \{1; 2; \ldots; 2009\}$, it is clear that

$\left\lfloor \dfrac{11x}{2010} \right\rfloor \in \{0; 1; \ldots; 10\}$.

$$X = \left\lfloor \frac{11}{2010} \right\rfloor + \left\lfloor \frac{11\times 2}{2010} \right\rfloor + \left\lfloor \frac{11\times 3}{2010} \right\rfloor + \left\lfloor \frac{11\times 4}{2010} \right\rfloor + \ldots + \left\lfloor \frac{11\times 2009}{2010} \right\rfloor = \sum_{\left\lfloor \frac{11x}{2010} \right\rfloor = 0} 0 + \sum_{\left\lfloor \frac{11x}{2010} \right\rfloor = 1} 1 + \ldots + \sum_{\left\lfloor \frac{11x}{2010} \right\rfloor = 10} 10 .$$

Let count the numbers of each case.

$2010 = 182\times 11 + 8$ so $\left\lfloor \dfrac{11x}{2010} \right\rfloor = 0$ for $1 \leq x \leq 182$.

$2010\times 2 = 2\times 182\times 11 + 2\times 8 = (2\times 182 + 1)\times 11 + 5$ so $\left\lfloor \dfrac{11x}{2010} \right\rfloor = 1$ for $182 + 1 \leq x \leq 2\times 182 + 1$. There are 183 x's suitable.

$2010\times 3 = (2\times 182 + 1)\times 11 + 5 + 182\times 11 + 8 = (3\times 182 + 2)\times 11 + 2$ so $\left\lfloor \dfrac{11x}{2010} \right\rfloor = 2$ for $2\times 182 + 2 \leq x \leq 3\times 182 + 2$. There are 183 x's suitable.

$2010\times 4 = (3\times 182 + 2)\times 11 + 2 + 182\times 11 + 8 = (4\times 182 + 2)\times 11 + 10$ so $\left\lfloor \dfrac{11x}{2010} \right\rfloor = 3$ for $3\times 182 + 3 \leq x \leq 4\times 182 + 2$. There are 182 x's suitable.

$2010\times 5 = (4\times 182 + 2)\times 11 + 10 + 182\times 11 + 8 = (5\times 182 + 3)\times 11 + 7$ so $\left\lfloor \dfrac{11x}{2010} \right\rfloor = 4$ for

$4\times 182+2 \le x \le 5\times 182+3$. There are 183 x's suitable.

$2010\times 6=(5\times 182+3)\times 11+7+182\times 11+8=(6\times 182+4)\times 11+4$ so $\left\lfloor \frac{11x}{2010} \right\rfloor = 5$ for

$5\times 182+4 \le x \le 6\times 182+4$. There are 183 x's suitable.

$2010\times 7=(6\times 182+4)\times 11+4+182\times 11+8=(7\times 182+5)\times 11+1$ so $\left\lfloor \frac{11x}{2010} \right\rfloor = 6$ for

$6\times 182+5 \le x \le 7\times 182+5$. There are 183 x's suitable.

$2010\times 8=(7\times 182+5)\times 11+1+182\times 11+8=(8\times 182+5)\times 11+9$ so $\left\lfloor \frac{11x}{2010} \right\rfloor = 7$ for

$7\times 182+6 \le x \le 8\times 182+5$. There are 182 x's suitable.

$2010\times 9=(8\times 182+5)\times 11+9+182\times 11+8=(9\times 182+6)\times 11+6$ so $\left\lfloor \frac{11x}{2010} \right\rfloor = 8$ for

$8\times 182+6 \le x \le 9\times 182+6$. There are 183 x's suitable.

$2010\times 10=(9\times 182+6)\times 11+6+182\times 11+8=(10\times 182+7)\times 11+3$ so $\left\lfloor \frac{11x}{2010} \right\rfloor = 9$ for

$9\times 182+7 \le x \le 10\times 182+7$. There are 183 x's suitable.

$2010\times 11=(10\times 182+7)\times 11+3+182\times 11+8=(11\times 182+8)\times 11+0$ so $\left\lfloor \frac{11x}{2010} \right\rfloor = 10$ for

$10\times 182+8 \le x \le 11\times 182+7$. There are 182 x's suitable. Let note that $11\times 182+7=2009$.

Finally

$X = 182\times 0+183\times 1+183\times 2+182\times 3+183\times 4+183\times 5+183\times 6+182\times 7+183\times 8+183\times 9+182\times 10=10\,045$.

Remark:

One can also remark that X is the number of grid points under the segment with equation $y=\frac{11}{2010}x$ with $0\le x\le 2009$.

This number is half the number of grid points inside the rectangle defined by the points (0, 0), (2010, 0), (11, 2010) and (0, 11), without the border, as there are no grid points on the line inside the rectangle.

21 Answer: 2007.

Let prove by induction that for $n\ge 3$ distinct numbers, for any arrangement, the number of *friendly* pairs is $n-3$.
For $n=3$, there are no *friendly* pairs as each number is neighbour to the others two. So the induction hypothesis is true for $n=3$.
Let assume that for n distinct numbers there are $n-3$ *friendly* pairs.
Consider $n+1$ distinct numbers. Let note N the largest one.
By deleting N, there are n numbers left.
The two numbers neighbours of N used to form a *friendly* but are not a *friendly* pair anymore after N is deleted.
All the others *frienfly* pairs remain *friendly* pairs after N is deleted, as N is the largest number.
And as N is the largest number, there was no *friendly* pair with N.
By induction hypothesis, there are $n-3$ *firendly* pairs with the n numbers left after N is deleted and only 1 *friendly* pair has been deleted by deleting N.
Therefore there are $(n-3)+1=(n+1)-3$ *friendly* pairs with $n+1$ distinct numbers.
The induction principle states that for all $n\ge 3$, there are $n-3$ *friendly* pairs with n distinct numbers.
Consequently there are 2007 *friendly* pairs with 2010 numbers.

22 Answer: 12345.

For $y=z=0$, we obtain $f(x^2)=x\,f(x)$ **(1)**.

Then for $x=0$ and $y=z$, $f(y\,f(y))=y\,f(y)$ **(2)**.

For $x = 0$, and $y = 1$, $f(1\,f(z)) = z\,f(1)$ so $f(f(x)) = f(1)\,x$ **(3)** for all real x. In particular, for $x = x^2$, $f(f(x^2)) = f(1)\,x^2$.
Then as $f(x^2) = x\,f(x)$ **(1)** and as $f(x\,f(x)) = x\,f(x)$ **(2)** we obtain $f(f(x^2)) = f(x\,f(x)) = x\,f(x)$.

It follows $x\,f(x) = f(1)\,x^2$ for all real x and for $x \neq 0$, $f(x) = f(1)\,x$.
Let's remark that $f(0) = 0$ so the equality holds for $x = 0$.
For all real x, $f(x) = f(1)\,x$.

Note that $f(f(1)) = f(1 \times f(1)) = 1 \times f(1) = f(1)$ and $f(f(1)) = f(1) \times f(1) = (f(1))^2$ so $(f(1))^2 = f(1)$ and then $f(1)\,(f(1) - 1) = 0$: or $f(1) = 0$ or $f(1) = 1$.
As f is a non-zero real valued function, $f(1) \neq 0$. In fact if $f(1) = 0$, $f(x) = 0 \times x$ for all real x.
In conclusion $f(x) = 1 \times x = x$ for all real x and $f(12\,345) = 12\,345$.

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23 Answer: 86422.

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24 Answer: 309.

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25 Answer: 2011.