Maths' Lab:

Singapore Mathematics Olympiad
Open Section 2010
Duration: 2h30

Instructions to contestants:
1. Answer All 25 questions.
2. Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.
3. No steps are needed to justify your answers.
4. Each question carries 1 mark.
5. No calculators are allowed.

1

Let S be the set of all integers n such that $\dfrac{8n^3 - 96n^2 + 360n - 400}{2n-7}$ is an integer. Find the value of $\sum_{n\in S}|n|$.

2

Determine the largest value of x for which $|x^2 - 4x - 39\,601| \geq |x^2 + 4x - 39\,601|$.

3

Given that $x = \lfloor 1^{1/3} \rfloor + \lfloor 2^{1/3} \rfloor + \lfloor 3^{1/3} \rfloor + \ldots + \lfloor 7999^{1/3} \rfloor$, find the value of $\left\lfloor \dfrac{x}{100} \right\rfloor$, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.
(For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 30 \rfloor = 30$, $\lfloor -10.5 \rfloor = -11$.)

4

Determine the smallest positive integer C such that $\dfrac{6^n}{n!} \leq C$ for all positive integers n .

5

Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1 perpendicular to CD at N with $AN > NB$.
A circle Γ_2 centred at C with radisu CN intersects Γ_1 at points P and Q, and the segments PQ and CD intersect at M.
Given that the radii of Γ_1 and Γ_2 are 61 and 60 respectively, find the length of AM .

6

Determine the minimum value of $\sum_{k=1}^{50} x_k$ where the summation is done over all possible positive numbers $x_1, \ldots, x_{50}$

satisfying $\sum_{k=1}^{50} \frac{1}{x_k} = 1$.

7

Find the sum of all positive integers p such that the expression $(x - p)(x - 13) + 4$ can be expressed in the form $(x + q)(x + r)$ for distinct integers q and r .

8

Let $p_k = 1 + \frac{1}{k} - \frac{1}{k^2} - \frac{1}{k^3}$, where k is a positive integer.

Find the least positive integer n such that the product $p_2 \, p_3 \ldots p_n$ exceeds 2010.

9

Let B be a point on a circle centred at O with diameter AC and let D and E be the circumcentres of the triangles OAB and OBC respectively.

Given that $\sin(\angle BOC) = \frac{4}{5}$ and $AC = 24$, find the area of the triangle BDE .

10

Let f be a real-valued function with rule $f(x) = x^3 + 3x^2 + 6x + 14$ defined for all real value of x.

It is given that a and b are two rea numbers such that $f(a) = 1$ and $f(b) = 19$.

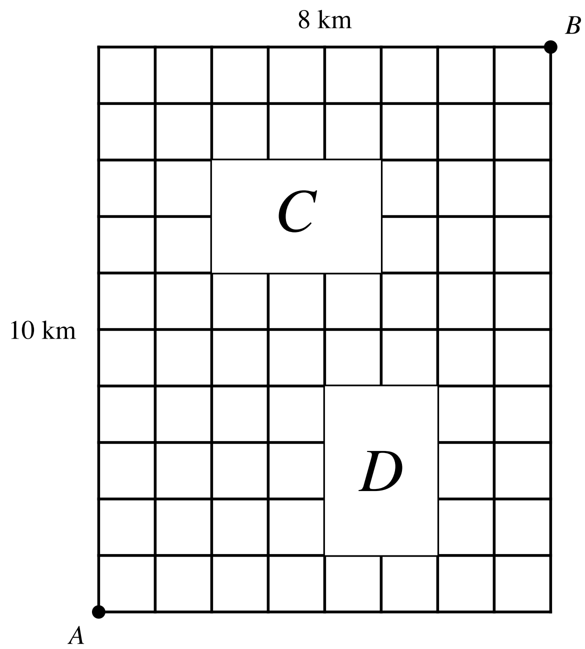
Find the value of $(a + b)^2$.

11

If $\cot(\alpha) + \cot(\beta) + \cot(\gamma) = -\frac{4}{5}$, $\tan(\alpha) + \tan(\beta) + \tan(\gamma) = \frac{17}{6}$ and $\cot(\alpha)\cot(\beta) + \cot(\beta)\cot(\gamma) + \cot(\gamma)\cot(\alpha) = -\frac{17}{5}$, find the value of $\tan(\alpha + \beta + \gamma)$.

12

The figure below shows a road map connecting two shopping malls A and B in a certain city.
Each side of the smallest square in the figure represents a road of distance 1 km.
Regions C and D represent two large residential estates in the town.
find the number of shortest routes to travel from A to B along the roads shown in the figure.



13

Let $a_1 = 1$, $a_2 = 2$ and for all $n \geq 2$, $a_{n+1} = \frac{2n}{n+1} a_n - \frac{n-1}{n+1} a_{n-1}$.
It is knowm that $a_n > 2 + \frac{2009}{2010}$ for all $n \geq m$, where m is a positive integer.
Find the least value of m .

14

It is known that $\sqrt{9 - 8\sin(50^\circ)} = a + b\sin(c^\circ)$ for eactly oneset of positive integers (a, b, c) where $0 < c < 90$. Find the value of $\frac{b+c}{a}$.

15

If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, find the value of $\lfloor \alpha^6 \rfloor$, where $\lfloor x \rfloor$ is the least positive integer not exceeding x .

16

If a positive integer cannot be written as the difference of two square numbers, then the integer is called a "cute" integer.

For example, 1, 2 and 4 are the first three "cute" integers.
Find the 2010^{th} cute integer.

(Note: A *square number* is a square of a positive integer. As an illustration, 1, 4, 9 and 16 are the first four square numbers.)

17

Let $f(x)$ be a polynomial in x of degrre 5. When $f(x)$ is divided by $x-1$, $x-2$, $x-3$, $x-4$ and x^2-x-1, $f(x)$ leaves a remainder of 3, 1, 7, 36 and $x-1$ respectively.
Find the square remainder when $f(x)$ is divided by $x+1$.

18

Determine the number of ordered pairs of positive integers (a, b) satisfying the equation $100(a+b) = ab - 100$.

19

Let $p = a^b + b^a$. If a, b and p are all prime, what is the value of p ?

20

Determine the value of the following expression:

$$\left\lfloor \frac{11}{2010} \right\rfloor + \left\lfloor \frac{11\times 2}{2010} \right\rfloor + \left\lfloor \frac{11\times 3}{2010} \right\rfloor + \left\lfloor \frac{11\times 4}{2010} \right\rfloor + \ldots + \left\lfloor \frac{11\times 2009}{2010} \right\rfloor$$

where $\lfloor y \rfloor$ denotes the greatest integer less or equal to y.

21

Numbers 1, 2, …, 2010 are placed on the circumference of a circle in some order.
The numbers i and j, where $i \neq j$ and $i, j \in \{1, 2, \ldots, 2010\}$ form a *friendly* pair if:

(i) i and j are not neighbours to each others and,

(ii) on one or both the arcs connecting i and j along the circle, all numbers in between are greater than both i and j.

Determine the minimal number of *friendly* pairs.

22

Let S be the set of all non-zero real valued functions f defined on the set of all real numbers such that $f(x^2 + y f(z)) = x f(x) + z f(y)$ for all numbers x, y and z.
Find the maximum value of $f(12\,345)$, where $f \in S$.

23

All possible 6-digit numbers, in each of which the digits occur in non-increasing order from left to right (e.g 966541), are written as a sequence in increasing order (the first three 6-digit numbers in this sequence are 100000, 110000, 111000 and so on).

If the 2010th number in this sequence is denoted by p, find the value of $\left\lfloor \frac{p}{10} \right\rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less or equal to x.

24

Find the number of permutations $a_1\, a_2\, a_3\, a_4\, a_5\, a_6$ of the six integers from 1 to 6 such that for all i from 1 to 5, a_{i+1} does not exceed a_i by 1.

25

Let $A = \left(\binom{2010}{0} - \binom{2010}{-1}\right)^2 + \left(\binom{2010}{1} - \binom{2010}{0}\right)^2 + \left(\binom{2010}{2} - \binom{2010}{1}\right)^2 + \ldots + \left(\binom{2010}{1005} - \binom{2010}{1004}\right)^2$.

Determine the minimum integer s such that $s\,A \geq \binom{4020}{2010}$.

(Note: for a given positive integer n, $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$ for $r = 0, 1, 2, 3, \ldots, n$; and for all other values of r, define $\binom{n}{r} = 0$.)