Maths Lab: Elements of solution

Singapore Mathematics Olympiad
Senior Section 2010
Duration: 2h30

Multiple Choice Questions

1 Answer (A).

Find the value of $((1\times2\times3) + (2\times4\times6) + (3\times6\times9) + \ldots + (335\times670\times1005))/((1\times3\times6) + (2\times6\times12) + (3\times9\times18) + \ldots + (335\times1005\times2010))$.

We have $((1\times2\times3) + (2\times4\times6) + (3\times6\times9) + \ldots + (335\times670\times1005))/((1\times3\times6) + (2\times6\times12) + (3\times9\times18) + \ldots + (335\times1005\times2010)) = \dfrac{(1\times2\times3)\left[1+2^3+3^3+\ldots+335^3\right]}{(1\times3\times6)\left[1+2^3+3^3+\ldots+335^3\right]}$

then $((1\times2\times3) + (2\times4\times6) + (3\times6\times9) + \ldots + (335\times670\times1005))/((1\times3\times6) + (2\times6\times12) + (3\times9\times18) + \ldots + (335\times1005\times2010)) = \dfrac{1}{3}$.

2 Answer (E).

Lets a, b, c and d be real numbers such that $\dfrac{b+c+d}{a} = \dfrac{a+c+d}{b} = \dfrac{a+b+d}{c} = \dfrac{a+b+c}{d} = r$.

As $r + 1 = 1 + \dfrac{b+c+d}{a} = \dfrac{a+b+c+d}{a}$ so $a(r+1) = a+b+c+d$.
Identically, $b(r+1) = c(r+1) = d(r+1) = a+b+c+d$ so $(r+1)(a+b+c+d) = 4(a+b+c+d)$.
Thus we obtain $(r-3)(a+b+c+d) = 0$.
Then $r = 3$ or $a+b+c+d = 0$.
Yet $a+b+c+d=0$ gives $-a = b+c+d$ so $\dfrac{b+c+d}{a} = \dfrac{-a}{a} = -1$ and identically $-b = a+c+d$, $-c = a+b+d$ and $-d = a+b+c$ so $\dfrac{a+c+d}{b} = \dfrac{a+b+d}{c} = \dfrac{a+b+c}{d} = -1$.
Finally, $r = 3$ or $r = -1$.

3 Answer (E).

If $0 < x < \dfrac{\pi}{2}$ and $\sin(x) - \cos(x) = \dfrac{\pi}{4}$ and $\tan(x) + \dfrac{1}{\tan(x)} = \dfrac{a}{b-\pi^c}$, where a, b and c are positive integers, find the value of $a+b+c$.

As $\sin(x) - \cos(x) = \dfrac{\pi}{4}$, then $(\sin(x)-\cos(x))^2 = \dfrac{\pi^2}{16}$.

Yet $(\sin(x)-\cos(x))^2 = \sin^2(x) - 2\sin(x)\cos(x) + \cos^2(x) = 1 - 2\sin(x)\cos(x)$ thus $\sin(x)\cos(x) = \dfrac{1}{2}\left(1 - \dfrac{\pi^2}{16}\right) = \dfrac{16-\pi^2}{32}$.

Then $\tan(x) + \dfrac{1}{\tan(x)} = \dfrac{\sin(x)}{\cos(x)} + \dfrac{\cos(x)}{\sin(x)} = \dfrac{\sin^2(x)+\cos^2(x)}{\sin(x)\cos(x)} = \dfrac{1}{\sin(x)\cos(x)} = \dfrac{1}{\frac{16-\pi^2}{32}} = \dfrac{32}{16-\pi^2}$.

Finally $a + b + c = 32 + 16 + 2 = 50$.

4 Answer (C).

The idea here is to decompose a cube in a difference of consecutive terms such that when adding the consecutive numbers 14, 15, …, 25, there will be simplifications.
Lets note that:

$$(n(n+1))^2 = n^4 + 2n^3 + n^2$$
$$(n(n-1))^2 = n^4 - 2n^3 + n^2$$

So $(n(n+1))^2 - (n(n-1))^2 = 4n^3$.
It follows:

$$14^3 = \frac{1}{4}\left(14^2\times 15^2 - 13^2\times 14^2\right)$$
$$15^3 = \frac{1}{4}\left(15^2\times 16^2 - 14^2\times 15^2\right)$$
$$\ldots$$
$$24^3 = \frac{1}{4}\left(24^2\times 25^2 - 23^2\times 24^2\right)$$
$$25^3 = \frac{1}{4}\left(25^2\times 26^2 - 24^2\times 25^2\right)$$

Then $14^3 + 15^3 + 16^3 + \ldots + 24^3 + 25^3 = \frac{1}{4}\left(25^2\times 26^2 - 13^2\times 14^2\right) = \frac{1}{4}(25\times 26 - 13\times 14)(25\times 26 + 13\times 14)$ and $14^3 + 15^3 + 16^3 + \ldots + 24^3 + 25^3 = \frac{1}{4}\times 26^2(25-7)(25+7) = \frac{1}{4}\times 26^2\times 9\times 2\times 16\times 2 = 3^2\times 4^2\times 26^2$.

We finally obtain $\sqrt{14^3 + 15^3 + 16^3 + \ldots + 24^3 + 25^3} = \sqrt{3^2\times 4^2\times 26^2} = 3\times 4\times 26 = 312$.

5 Answer (B).

As ABC is isosceles, we can deduce that E is the midpoint of $[BC]$ so $BE = EC = \sqrt{5}$ and that BC is perpendicular to
AD.

Then as BD is parallel to FC, the Thales' Theorem gives $\frac{BE}{CE} = \frac{EF}{ED}$ so $EF = ED$.

Therefore, as F is the midpoint of OE, we have $OF = FE = ED = \frac{1}{3}OD = \frac{1}{3}r$ where r is the radius of the circle.

As the triangle OEB is rectangle at E, the Pythagoreas' Theorem gives $r^2 = \left(\frac{2}{3}r\right)^2 + \left(\sqrt{5}\right)^2$ and so $r^2 = 9$ and $r = 3$.

Again in the thriangle EDC rectangle at E, $CD^2 = \left(\frac{1}{3}r\right)^2 + \sqrt{5} = 1 + 5 = 6$ so $CD = \sqrt{6}$ cm.

6 Answer (E).

We know that exists an integer n such as $y = n^2$ so $n^2 = (x-90)^2 - 4907$ thus $((x-90)-n)((x-90)+n) = 4907$ id est $(k-n)(k+n) = 4907$ with $k^2 = (x-90)^2$ and $k > 0$.
As $4907 = 7\times 701$ with 701 prime number.
(In fact as $25^2 < 701 < 30^2$, we know that if 701 wasn't a prime number it will be divisible by 2,3,5,7,11,13,17,19,23 or 29 and that's not the case.)

Therefore we obtain:

- $k - m = 1$ and $k + m = 4907$, then $k = 2454$ and $n = 2453$
- $k - m = 4907$ and $k + m = 1$, then $k = 2454$ and $n = -2453$
- $k - m = 7$ and $k + m = 701$, then $k = 354$ and $n = 347$
- $k - m = 701$ and $k + m = 7$, then $k = 354$ and $n = -347$.

As $y = n^2$, we deduce $y = 2453^2$ or $y = 347^2$.

Moreover

- or $x - 90 = 2454$ so $x = 2544$ or $90 - x = 2454$ so $x = -2364$: we have 2 ordered pairs $\left(-2364,\ 2453^2\right)$ and $\left(2544,\ 2453^2\right)$,
- or $x - 90 = 354$ so $x = 444$ or $90 - x = 354$ so $x = -264$: we have 2 ordered pairs $\left(-264,\ 347^2\right)$ and $\left(354,\ 347^2\right)$.

7 Answer (D).

Lets describe the cases of non-empty subsets verifying the condition "Good":

- the singletons: only composed by an even number so 5 subsets;
- the pairs: two even numbers or 1 even and 1 odd, so $\binom{5}{2} + \binom{5}{1}\binom{5}{1} = 10 + 25 = 35$ subsets;
- the 3–lists: two even and one odd or three even, so $\binom{5}{2}\binom{5}{1} + \binom{5}{3} = 10 \times 5 + 10 = 60$ subsets;
- the 4–lists: 2 even and 2 odd, 3 even and 1 odd or 4 even, so $\binom{5}{2}\binom{5}{2} + \binom{5}{3}\binom{5}{1} + \binom{5}{4} = 100 + 50 + 5 = 155$ subsets;
- the 5–lists: 3 even and 2 odd, 4 even and 1 odd or 5 even, so $\binom{5}{3}\binom{5}{2} + \binom{5}{4}\binom{5}{1} + \binom{5}{5} = 100 + 25 + 1 = 126$ subsets;
- the 6–lists: 3 even and 3 odd, 4 even and 2 odd or 5 even and 1 odd, so $\binom{5}{3}\binom{5}{3} + \binom{5}{4}\binom{5}{2} + \binom{5}{5}\binom{5}{1} = 100 + 50 + 5 = 155$ subsets;
- the 7–lists: 4 even and 3 odd or 5 even and 2 odd, so $\binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = 50 + 10 = 60$ subsets;
- the 8–lists: 4 even and 4 odd or 5 even and 3 odd, so $\binom{5}{4}\binom{5}{4} + \binom{5}{5}\binom{5}{3} = 25 + 10 = 35$ subsets;
- the 9–lists: 5 even and 4 odd so $\binom{5}{5}\binom{5}{4} = 5$ subsets;
- the 10–lists: 5 even and 5 odd so 1 subset.

the overall number of subsets is then $5 + 35 + 60 + 155 + 126 + 155 + 60 + 35 + 5 + 1 = 637$ subsets.

8 Answer (B).

As $f(r) = k$ and $f(s) = k$, then r and s are symetric regarding the x-coordinate of the summit of the quadratic function, it is to say $-\dfrac{b}{2a}$.

We then deduce $s + r = 2 \times \left(-\dfrac{b}{2a}\right) = -\dfrac{b}{a}$.

Thus $f(r + s) = a \times \left(-\dfrac{b}{a}\right)^2 + b \times \left(-\dfrac{b}{a}\right) + c = \dfrac{b^2}{a} - \dfrac{b^2}{a} + c = c$.

9 Answer (D).

Find the number of positive integers $k < 100$ such that $2\,(3^{6\,n}) + k(2^{3\,n+1}) - 1$ is divisible by 7 for any positive integer n .

Note that $2\,(3^{6\,n}) + k(2^{3\,n+1}) - 1 = 2\times(3^6)^n + 2\,k\times(2^3)^n - 1 = 2\,((3^6)^n + k\times(2^3)^n) - 1$.

We have $2^3 = 8 = 7 + 1$ so the remainder of 2^3 in the division by 7 is 1. Thus the remainder of $2^{3\,n}$ is 1 too.

Identically, $3^6 = (3^2)^3$ so the remainder of 3^6 in the division by 7 is 1, thus the remainder of $3^{6\,n}$ is 1.

Finally the remainder of $((3^6)^n + k\times(2^3)^n)$ is $1 + k\times 1 = k + 1$.

We have to determine the number of positive integers $k < 100$ such that $2\,(k + 1) - 1 = 2\,k + 1$ is divisible by 7.

So it must exist integer p such that $2\,k + 1 = 7\,p$, $2\,k = 7\,p - 1 = 7\,(p - 1) + 6$ and we deduce that $k = 7\,p' + 3$.

As $0 \le 7\,p' + 3 < 100$, $0 \le p' < 13$.

We obtain 14 possibilities.

10 Answer (C).

Let $ABCD$ be a trapezium with AD parallel to BC and $\angle ADC = 90\,°$, as shown in the figure below. Given that M is the midpoint of AB with $CM = \frac{13}{2}$ cm and $BC + CD + DA = 17$ cm, find the area of the trapezium $ABCD$ in cm^2.

Lets consider point E, intersection of the lines AD et CM.

The angles $\angle AME$ and $\angle BME$ are opposite angles they are congruent.

As lines AD and CB are parallel and line CM is a secant then the angles $\angle AEM$ and $\angle BCM$ are alternate angles and congruent.

Moreover $AM = ME$ as M is the midpoint of AB .

Theregore the triangles AEM and CBM are congruent.

We deduce $AE = CB$ and $ME = CM$ so $AE + AD + DC = 17$ and $CE = 13$ and clearly the area of the trapezium is equal to the area of triangle CDE.

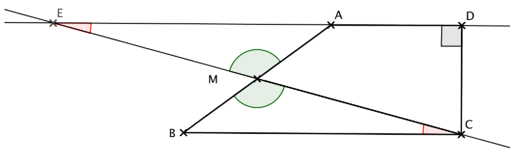
The triangle EDC is rectangle at point D, according to the Pythagoreas' theorem, $CE^2 = CD^2 + DE^2$.

So $13^2 = CD^2 + (17 - DC)^2$ so DC is solution of the equation $2\,x^2 - 34\,x + 120 = 0$.

The roots of this quadratic equation are 5 and 12.

(The discriminant is $\Delta = 196 > 0$ so the roots are $\frac{34 - \sqrt{196}}{2\times 2} = 5$ and $\frac{34 + \sqrt{196}}{2\times 2} = 12$).

Finally $DE = 12$ or $DE = 5$ so the area of the triangle is in both cases $\frac{5\times 12}{2} = 30$.



Remark:

Let's note S the area of trapezium $ABCD$.

Then $S = \frac{1}{2}\,DE\times DC$ as previously explained.

Now $(DE + DC)^2 = DE^2 + 2\,DE\times DC + DC^2 = 4\,S + CE^2$. In fact, Pythagoreas' Theorem in triangle DEC right-angled at D, shows that $CE^2 = DE^2 + DC^2$.

Finally as $DE + DC = AD + CD + CB = 17$, we deduce $4\,S + 13^2 = 17^2$ and so $S = 30$.