Maths Lab: Wordings

Maths Lab: Elements of solution

Singapore Mathematics Olympiad
Senior Section 2010
Duration: 2 h30.

The number of ◊ indicates the relativ difficulty of the question.

Multiple Choice Questions

1 Answer (A).

Find the value of $\dfrac{(1\times2\times3)+(2\times4\times6)+(3\times6\times9)+\ldots+(335\times670\times1005)}{(1\times3\times6)+(2\times6\times12)+(3\times9\times18)+\ldots+(335\times1005\times2010)}$.

We have $\dfrac{(1\times2\times3)+(2\times4\times6)+(3\times6\times9)+\ldots+(335\times670\times1005)}{(1\times3\times6)+(2\times6\times12)+(3\times9\times18)+\ldots+(335\times1005\times2010)} = \dfrac{(1\times2\times3)\left[1+2^3+3^3+\ldots+335^3\right]}{(1\times3\times6)\left[1+2^3+3^3+\ldots+335^3\right]}$ then $\dfrac{(1\times2\times3)+(2\times4\times6)+(3\times6\times9)+\ldots+(335\times670\times1005)}{(1\times3\times6)+(2\times6\times12)+(3\times9\times18)+\ldots+(335\times1005\times2010)} = \dfrac{1}{3}$.

2 Answer (E).

Lets a, b, c and d be real numbers such that $\dfrac{b+c+d}{a} = \dfrac{a+c+d}{b} = \dfrac{a+b+d}{c} = \dfrac{a+b+c}{d} = r$.

As $r+1 = 1 + \dfrac{b+c+d}{a} = \dfrac{a+b+c+d}{a}$ so $a(r+1) = a+b+c+d$.
Identically, $b(r+1) = c(r+1) = d(r+1) = a+b+c+d$ so $(r+1)(a+b+c+d) = 4(a+b+c+d)$.
Thus we obtain $(r-3)(a+b+c+d) = 0$.
Then $r = 3$ or $a+b+c+d = 0$.
Yet $a+b+c+d = 0$ gives $-a = b+c+d$ so $\dfrac{b+c+d}{a} = \dfrac{-a}{a} = -1$ and identically $-b = a+c+d$, $-c = a+b+d$ and $-d = a+b+c$ so $\dfrac{a+c+d}{b} = \dfrac{a+b+d}{c} = \dfrac{a+b+c}{d} = -1$.
Finally, $r = 3$ or $r = -1$.

3 ◊◊ Answer (E).

If $0 < x < \dfrac{\pi}{2}$ and $\sin(x) - \cos(x) = \dfrac{\pi}{4}$ and $\tan(x) + \dfrac{1}{\tan(x)} = \dfrac{a}{b-\pi^c}$, where a, b and c are positive integers, find the value of $a+b+c$.

As $\sin(x) - \cos(x) = \dfrac{\pi}{4}$, then $(\sin(x)-\cos(x))^2 = \dfrac{\pi^2}{16}$.

Yet $(\sin(x)-\cos(x))^2 = \sin^2(x) - 2\sin(x)\cos(x) + \cos^2(x) = 1 - 2\sin(x)\cos(x)$ thus $\sin(x)\cos(x) = \dfrac{1}{2}\left(1 - \dfrac{\pi^2}{16}\right) = \dfrac{16-\pi^2}{32}$.

Then $\tan(x) + \dfrac{1}{\tan(x)} = \dfrac{\sin(x)}{\cos(x)} + \dfrac{\cos(x)}{\sin(x)} = \dfrac{\sin^2(x)+\cos^2(x)}{\sin(x)\cos(x)} = \dfrac{1}{\sin(x)\cos(x)} = \dfrac{1}{\frac{16-\pi^2}{32}} = \dfrac{32}{16-\pi^2}$.

Finally $a + b + c = 32 + 16 + 2 = 50$.

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4 ◊ **Answer (C).**

The idea here is to decompose a cube in a difference of consecutive terms such that when adding the consecutive numbers 14, 15, …, 25, there will be simplifications.
Lets note that:

$$(n(n+1))^2 = n^4 + 2n^3 + n^2$$
$$(n(n-1))^2 = n^4 - 2n^3 + n^2$$

So $(n(n+1))^2 - (n(n-1))^2 = 4n^3$.
It follows:

$$14^3 = \frac{1}{4}\left(14^2\times15^2 - 13^2\times14^2\right)$$
$$15^3 = \frac{1}{4}\left(15^2\times16^2 - 14^2\times15^2\right)$$
$$\ldots$$
$$24^3 = \frac{1}{4}\left(24^2\times25^2 - 23^2\times24^2\right)$$
$$25^3 = \frac{1}{4}\left(25^2\times26^2 - 24^2\times25^2\right)$$

Then $14^3 + 15^3 + 16^3 + \ldots + 24^3 + 25^3 = \frac{1}{4}\left(25^2\times26^2 - 13^2\times14^2\right) = \frac{1}{4}(25\times26 - 13\times14)(25\times26 + 13\times14)$ and

$14^3 + 15^3 + 16^3 + \ldots + 24^3 + 25^3 = \frac{1}{4}\times26^2(25-7)(25+7) = \frac{1}{4}\times26^2\times9\times2\times16\times2 = 3^2\times4^2\times26^2$.

We finally obtain $\sqrt{14^3 + 15^3 + 16^3 + \ldots + 24^3 + 25^3} = \sqrt{3^2\times4^2\times26^2} = 3\times4\times26 = 312$.

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5 **Answer (B).**

As ABC is isosceles, we can deduce that E is the midpoint of $[BC]$ so $BE = EC = \sqrt{5}$ and that BC is perpendicular to
AD.

Then as BD is parallel to FC, the Thales' Theorem gives $\frac{BE}{CE} = \frac{EF}{ED}$ so $EF = ED$.

Therefore, as F is the midpoint of OE, we have $OF = FE = ED = \frac{1}{3}OD = \frac{1}{3}r$ where r is the radius of the circle.

As the triangle OEB is rectangle at E, the Pythagoreas' Theorem gives $r^2 = \left(\frac{2}{3}r\right)^2 + \left(\sqrt{5}\right)^2$ and so $r^2 = 9$ and $r = 3$.

Again in the thriangle EDC rectangle at E, $CD^2 = \left(\frac{1}{3}r\right)^2 + \sqrt{5} = 1 + 5 = 6$ so $CD = \sqrt{6}$ cm.

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6 ◊ **Answer (E).**

We know that exists an integer n such as $y = n^2$ so $n^2 = (x-90)^2 - 4907$ thus $((x-90)-n)((x-90)+n) = 4907$ id est $(k-n)(k+n) = 4907$ with $k^2 = (x-90)^2$ and $k > 0$.
As $4907 = 7\times701$ with 701 prime number.
(In fact as $25^2 < 701 < 30^2$, we know that if 701 wasn't a prime number it will be divisible by 2,3,5,7,11,13,17,19,23 or 29 and that's not the case.)

Therefore we obtain:

- $k - m = 1$ and $k + m = 4907$, then $k = 2454$ and $n = 2453$
- $k - m = 4907$ and $k + m = 1$, then $k = 2454$ and $n = -2453$
- $k - m = 7$ and $k + m = 701$, then $k = 354$ and $n = 347$
- $k - m = 701$ and $k + m = 7$, then $k = 354$ and $n = -347$.

As $y = n^2$, we deduce $y = 2453^2$ or $y = 347^2$.

Moreover

- or $x - 90 = 2454$ so $x = 2544$ or $90 - x = 2454$ so $x = -2364$: we have 2 ordered pairs $\left(-2364,\, 2453^2\right)$ and $\left(2544,\, 2453^2\right)$,
- or $x - 90 = 354$ so $x = 444$ or $90 - x = 354$ so $x = -264$: we have 2 ordered pairs $\left(-264,\, 347^2\right)$ and $\left(354,\, 347^2\right)$.

7 Answer (D).

Lets describe the cases of non-empty subsets verifying the condition "Good":

- the singletons: only composed by an even number so 5 subsets;
- the pairs: two even numbers or 1 even and 1 odd, so $\binom{5}{2} + \binom{5}{1}\binom{5}{1} = 10 + 25 = 35$ subsets;
- the 3–lists: two even and one odd or three even, so $\binom{5}{2}\binom{5}{1} + \binom{5}{3} = 10\times 5 + 10 = 60$ subsets;
- the 4–lists: 2 even and 2 odd, 3 even and 1 odd or 4 even, so $\binom{5}{2}\binom{5}{2} + \binom{5}{3}\binom{5}{1} + \binom{5}{4} = 100 + 50 + 5 = 155$ subsets;
- the 5–lists: 3 even and 2 odd, 4 even and 1 odd or 5 even, so $\binom{5}{3}\binom{5}{2} + \binom{5}{4}\binom{5}{1} + \binom{5}{5} = 100 + 25 + 1 = 126$ subsets;
- the 6–lists: 3 even and 3 odd, 4 even and 2 odd or 5 even and 1 odd, so $\binom{5}{3}\binom{5}{3} + \binom{5}{4}\binom{5}{2} + \binom{5}{5}\binom{5}{1} = 100 + 50 + 5 = 155$ subsets;
- the 7–lists: 4 even and 3 odd or 5 even and 2 odd, so $\binom{5}{4}\binom{5}{3} + \binom{5}{5}\binom{5}{2} = 50 + 10 = 60$ subsets;
- the 8–lists: 4 even and 4 odd or 5 even and 3 odd, so $\binom{5}{4}\binom{5}{4} + \binom{5}{5}\binom{5}{3} = 25 + 10 = 35$ subsets;
- the 9–lists: 5 even and 4 odd so $\binom{5}{5}\binom{5}{4} = 5$ subsets;
- the 10–lists: 5 even and 5 odd so 1 subset.

the overall number of subsets is then $5 + 35 + 60 + 155 + 126 + 155 + 60 + 35 + 5 + 1 = 637$ subsets.

8 Answer (B).

As $f(r) = k$ and $f(s) = k$, then r and s are symetric regarding the x-coordinate of the summit of the quadratic function, it is to say $-\dfrac{b}{2a}$.

We then deduce $s + r = 2\times\left(-\dfrac{b}{2a}\right) = -\dfrac{b}{a}$.

Thus $f(r + s) = a\times\left(-\dfrac{b}{a}\right)^2 + b\times\left(-\dfrac{b}{a}\right) + c = \dfrac{b^2}{a} - \dfrac{b^2}{a} + c = c$.

9 ◊◊ Answer (D).

Find the number of positive integers $k < 100$ such that $2(3^{6n}) + k(2^{3n+1}) - 1$ is divisible by 7 for any positive integer n .

Note that $2(3^{6n}) + k(2^{3n+1}) - 1 = 2\times(3^6)^n + 2k\times(2^3)^n - 1 = 2((3^6)^n + k\times(2^3)^n) - 1$.

We have $2^3 = 8 = 7 + 1$ so the remainder of 2^3 in the division by 7 is 1. Thus the remainder of 2^{3n} is 1 too.

Identically, $3^6 = (3^2)^3$ so the remainder of 3^6 in the division by 7 is 1, thus the remainder of 3^{6n} is 1.

Finally the remainder of $((3^6)^n + k\times(2^3)^n)$ is $1 + k\times1 = k + 1$.

We have to determine the number of positive integers $k < 100$ such that $2(k + 1) - 1 = 2k + 1$ is divisible by 7.

So it must exist integer p such that $2k + 1 = 7p$, $2k = 7p - 1 = 7(p - 1) + 6$ and we deduce that $k = 7p' + 3$.

As $0 \le 7p' + 3 < 100$, $0 \le p' < 13$.

We obtain 14 possibilities.

10 Answer (C).

Lets consider point E, intersection of the lines AD et CM.

The angles $\angle AME$ and $\angle BME$ are opposite angles they are congruent.

As lines AD and CB are parallel and line CM is a secant then the angles $\angle AEM$ and $\angle BCM$ are alternate angles and congruent.

Moreover $AM = ME$ as M is the midpoint of AB .

Theregore the triangles AEM and CBM are congruent.

We deduce $AE = CB$ and $ME = CM$ so $AE + AD + DC = 17$ and $CE = 13$ and clearly the area of the trapezium is equal to the area of triangle CDE.

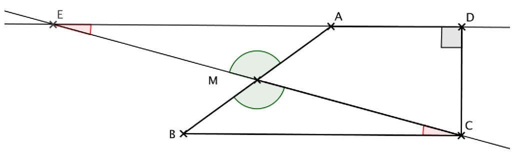
The triangle EDC is rectangle at point D, according to the Pythagoreas' theorem, $CE^2 = CD^2 + DE^2$.

So $13^2 = CD^2 + (17 - DC)^2$ so DC is solution of the equation $2x^2 - 34x + 120 = 0$.

The roots of this quadratic equation are 5 and 12.

(The discriminant is $\Delta = 196 > 0$ so the roots are $\dfrac{34 - \sqrt{196}}{2\times2} = 5$ and $\dfrac{34 + \sqrt{196}}{2\times2} = 12$).

Finally $DE = 12$ or $DE = 5$ so the area of the triangle is in both cases $\dfrac{5\times12}{2} = 30$.



Remark:

Let's note S the area of trapezium $ABCD$.

Then $S = \dfrac{1}{2} DE\times DC$ as previously explained.

Now $(DE + DC)^2 = DE^2 + 2DE\times DC + DC^2 = 4S + CE^2$. In fact, Pythagoreas' Theorem in triangle DEC right-angled at D, shows that $CE^2 = DE^2 + DC^2$.

Finally as $DE + DC = AD + CD + CB = 17$, we deduce $4S + 13^2 = 17^2$ and so $S = 30$.

Short Questions

11 Answer: $x = 132$.

Denote L et w the dimensions of the original rectangle.

Then the area is given by $A = L\times w$.

We then know that $(L+6)\times(w-2)=L\times w$ so we obtain $-2L+6w-12=0$, $L-3w+6=0$.
We also know that $(L-12)\times(w+6)=L\times w$ so we also obtain $6L-12w-72=0$, $L-2w-12=0$.
So the dimensions of the original rectangle are the couple solution of the system $\begin{cases} L-3w+6=0 \\ L-2w-12=0 \end{cases}$.
Thus $L=48$ and $w=18$ so $x=2(L+w)=132$.

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12 ◊ Answer: 2575.

For all integer r, $u_r = 1+2+\ldots+r = \dfrac{r(r+1)}{2}$.

Thus for all integer $j \geq 1$, $\displaystyle\sum_{i=1}^{j}\frac{1}{u_i} = \sum_{i=1}^{j}\frac{2}{i(i+1)}$.

Let's remind that $\dfrac{1}{i}-\dfrac{1}{i+1}=\dfrac{i+1-i}{i(i+1)}=\dfrac{1}{i(i+1)}$, so $\displaystyle\sum_{i=1}^{j}\frac{1}{u_i}=2\sum_{i=1}^{j}\left(\frac{1}{i}-\frac{1}{i+1}\right)=2\left(1-\frac{1}{j+1}\right)=\frac{2j}{j+1}$.

Then $\dfrac{j}{\sum_{i=1}^{j}\frac{1}{u_i}}=\dfrac{j}{\frac{2j}{j+1}}=\dfrac{j+1}{2}$, so

$$\frac{1}{\left(\frac{1}{u_1}\right)}+\frac{2}{\left(\frac{1}{u_1}+\frac{1}{u_2}\right)}+\frac{3}{\left(\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right)}+\ldots+\frac{100}{\left(\frac{1}{u_1}+\frac{1}{u_2}+\ldots+\frac{1}{u_{100}}\right)}=\frac{1}{2}\sum_{i=1}^{100}(j+1).$$

Yet, $\displaystyle\sum_{i=1}^{100}(j+1)=\sum_{i=2}^{101}j=\frac{(2+101)\times 100}{2}=5150$ so the value of

$\dfrac{1}{\left(\frac{1}{u_1}\right)}+\dfrac{2}{\left(\frac{1}{u_1}+\frac{1}{u_2}\right)}+\dfrac{3}{\left(\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right)}+\ldots+\dfrac{100}{\left(\frac{1}{u_1}+\frac{1}{u_2}+\ldots+\frac{1}{u_{100}}\right)}$ is 2575 .

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13 Answer: $k = 501$.

As $2010!=1\times2\times\ldots\times2010$ and as $10=2\times5$, so it is enough to count how many 5's in $2010!$.
We have $2010=5\times402$ so there are 402 multiples of 5 in the set $\{1;2;\ldots;2010\}$ and then at least 402 factors 5's in the product $2010!$.
Identically, $2010=5^2\times80+10$ so there are 80 multiples of 5^2 in the set $\{1;2;\ldots;2010\}$, $2010=5^3\times16+10$ so there are 16 multiples of 5^3 in the set $\{1;2;\ldots;2010\}$ and finally $2010=5^4\times3+135$ so there are 3 multiples of 5^4 in the set $\{1;2;\ldots;2010\}$.
As each multiples of 5^2 is counted in the 402 multiples of 5, and so on, the number of factor 5 in the product $2010!$ is given by $402+80+16+3=501$.
So we can assure that $2010!=5^{501}\times M'$ with M' an integer not divisible by 5.
Clearly there are at list 501 factors 2 in the product $2010!$ and so $2010!=10^{501}\times M$ with M an integer not divisible by 10.

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14 ◊◊ Answer: 35.

If $a>b>1$ and $\dfrac{1}{\log_a(b)}+\dfrac{1}{\log_b(a)}=\sqrt{1229}$, find the value of $\dfrac{1}{\log_{ab}(b)}-\dfrac{1}{\log_{ab}(a)}$.

As $a>b>1$, $\dfrac{1}{\log_{ab}(b)}-\dfrac{1}{\log_{ab}(a)}>0$.

Then $\dfrac{1}{\log_{a\,b}(b)} - \dfrac{1}{\log_{a\,b}(a)} = \dfrac{1}{\frac{\ln(b)}{\ln(a\,b)}} - \dfrac{1}{\frac{\ln(a)}{\ln(a\,b)}} = \dfrac{\ln(a\,b)}{\ln(b)} - \dfrac{\ln(a\,b)}{\ln(a)} = \dfrac{\ln(a)+\ln(b)}{\ln(b)} - \dfrac{\ln(a)+\ln(b)}{\ln(a)} = \dfrac{\ln(a)}{\ln(b)} - \dfrac{\ln(b)}{\ln(a)}$.

As $\dfrac{\ln(a)}{\ln(b)} - \dfrac{\ln(b)}{\ln(a)} > 0$, $\dfrac{1}{\log_{a\,b}(b)} - \dfrac{1}{\log_{a\,b}(a)} = \sqrt{\left(\dfrac{\ln(a)}{\ln(b)} - \dfrac{\ln(b)}{\ln(a)}\right)^2}$.

Lets yet note $\dfrac{1}{\log_a(b)} + \dfrac{1}{\log_b(a)} = \dfrac{1}{\frac{\ln(b)}{\ln(a)}} + \dfrac{1}{\frac{\ln(a)}{\ln(b)}} = \dfrac{\ln(a)}{\ln(b)} + \dfrac{\ln(b)}{\ln(a)}$ so

$\left(\dfrac{\ln(a)}{\ln(b)} - \dfrac{\ln(b)}{\ln(a)}\right)^2 = \left(\dfrac{\ln(a)}{\ln(b)} + \dfrac{\ln(b)}{\ln(a)}\right)^2 - 4\dfrac{\ln(a)}{\ln(b)} \times \dfrac{\ln(b)}{\ln(a)} = 1229 - 4 = 1225.$

Finally $\dfrac{1}{\log_{a\,b}(b)} - \dfrac{1}{\log_{a\,b}(a)} = \sqrt{1225} = 35.$

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15 ◊ Answer: 1994.

As $\left\lfloor \dfrac{2010}{k} \right\rfloor \le \dfrac{2010}{k} < \left\lfloor \dfrac{2010}{k} \right\rfloor + 1$ then $0 \le \dfrac{2010}{k} - \left\lfloor \dfrac{2010}{k} \right\rfloor < 1$ and we obtain $\left\lceil \dfrac{2010}{k} - \left\lfloor \dfrac{2010}{k} \right\rfloor \right\rceil = 0$ if $\dfrac{2010}{k} = \left\lfloor \dfrac{2010}{k} \right\rfloor$ and 1 otherwise.

Yet $\dfrac{2010}{k} = \left\lfloor \dfrac{2010}{k} \right\rfloor$ if $\dfrac{2010}{k} \in \mathbb{N}$ so if k is a divisor of 2010.

As $2010 = 2\times3\times5\times67$, 2010 has $2^4 = 16$ positive divisors.

Therefore $\displaystyle\sum_{i=1}^{2010} \left\lceil \dfrac{2010}{k} - \left\lfloor \dfrac{2010}{k} \right\rfloor \right\rceil = 2010 - 16 = 1994.$

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16 ◊ Answer: 1005.

Lets note that for $i = 1$ to 2010, $f\left(\dfrac{i}{2011}\right) = \dfrac{\left(\frac{i}{2011}\right)^{2010}}{\left(\frac{i}{2011}\right)^{2010} + \left(1 - \frac{i}{2011}\right)^{2010}}$ and

$f\left(\dfrac{2011-i}{2011}\right) = \dfrac{\left(\frac{2011-i}{2011}\right)^{2010}}{\left(\frac{2011-i}{2011}\right)^{2010} + \left(1 - \frac{2011-i}{2011}\right)^{2010}} = \dfrac{\left(1 - \frac{i}{2011}\right)^{2010}}{\left(1 - \frac{i}{2011}\right)^{2010} + \left(\frac{i}{2011}\right)^{2010}}$.

Therefore $f\left(\frac{i}{2011}\right) + f\left(\frac{2011-i}{2011}\right) = 1$.

Consequently $f\left(\dfrac{1}{2011}\right) + f\left(\dfrac{2}{2011}\right) + f\left(\dfrac{3}{2011}\right) + \ldots + f\left(\dfrac{2010}{2011}\right) = \displaystyle\sum_{i=1}^{1005} 1 = 1005.$

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17 Answer: 216.

If a, b and c are positive real numbers such that $a\,b + a + b = b\,c + b + c = c\,a + c + a = 35$, find the value of $(a+1)(b+1)(c+1)$.

Lets note that as $(a+1)(b+1) = a\,b + a + b + 1$, $(a+1)(b+1) = 35 + 1 = 36$.

Identically, $(a+1)(c+1) = 36$ and $(b+1)(c+1) = 36$.

Therefore $(a+1)(b+1)(c+1) = 36\,(a+1) = 36\,(b+1) = 36\,(c+1)$.

Then $a+1 = b+1 = c+1$ so $(a+1)^2 = 36$ then $a+1 = b+1 = c+1 = 6$ so $(a+1)(b+1)(c+1) = 6^3 = 216$.

18 ◊◊ Answer: 27.

As AB and CD are parallel chords, and as $OA = OB$ and $OC = OD$ (the triangles OAB and OCD are isosceles at O, the perpendicular bissector of $[AB]$ is also the perpendicular bissector of $[CD]$. It goes through O.
Let's note I and J the midpoints of $[AB]$ and $[CD]$ respectively.

Then $AI = \dfrac{AB}{2} = 23$ and $CJ = \dfrac{CD}{2} = 9$.

Moreover $\angle IOA = \frac{1}{2}\angle BOA$ and $\angle JOC = \frac{1}{2}\angle DOC$ so as $\angle AOB = 3\times\angle COD$, $\angle IOA = 3\times\angle IOC = 3x$ with $x = \angle IOC$.

The triangles OIA and OJC are rectangle at I and J respectively so we obtain $\sin(\angle IOA) = \dfrac{IA}{OA}$ namely $\sin(3x) = \dfrac{IA}{r}$ and identically, $\sin(x) = \dfrac{CJ}{r}$.

Then as $\dfrac{AI}{CJ} = \dfrac{23}{9}$, $\dfrac{r\sin(3x)}{r\sin(x)} = \dfrac{23}{9}$.

For all x, $\sin(3x) = \sin(2x + x) = \sin(2x)\cos(x) + \sin(x)\cos(2x)$ then $\sin(2x) = 2\cos(x)\sin(x)$ and $\cos(2x) = 1 - 2\sin^2(x)$ so $\sin(3x) = 2\cos(x)\sin(x)\cos(x) + \sin(x)\left(1 - 2\sin^2(x)\right)$ and finally $\sin(3x) = 2\sin(x)\left(1 - \sin^2(x)\right) + \sin(x) - 2\sin^3(x) = 3\sin(x) - 4\sin^3(x)$.

We obtain $\dfrac{\sin(3x)}{\sin(x)} = \dfrac{3\sin(x) - 4\sin^3(x)}{\sin(x)} = 3 - 4\sin^2(x)$ and so $3 - 4\sin^2(x) = \frac{23}{9}$.

We deduce $\sin^2(x) = \dfrac{\left(3 - \frac{23}{9}\right)}{4} = \dfrac{1}{9}$.

As $0 < x < 180°$, $\sin(x) > 0$ and so $\sin(x) = \dfrac{1}{3}$.

It follows $r = \dfrac{CN}{\sin(x)} = \dfrac{9}{\frac{1}{3}} = 27$.

19 ◊◊ Answer: 2010.

Find the number of ways that 2010 can be written as a sum of one or more positive integers in non-decreasing order such that the difference between the last term and the first term is at most 1.
First lets note that as we consider a sum of one or more positive integers in non-decreasing order such that the difference between the last term and the first term is at most 1, the sum can be written $s = (a + \ldots + a) + ((a+1) + \ldots + (a+1)) = k\,a + k'(a+1)$ with $k, k' \in \mathbb{N}$ and $a \in [\![1; 2010]\!]$.
Second, we know that for each $a \in [\![1; 2010]\!]$, there exists an unique couple $(q; r)$ such that $2010 = q\,a + r$ and $0 \le r < a$.
Then $2010 = (q - r)\,a + r(a+1)$.
Thus 2010 is written as the sum of $q - r$ copies of a and r copies of $(a+1)$.
For each $a \in [\![1; 2010]\!]$, there is only one way to write 2010 this way as the couple given by the division algorithm is unique.
Finally we obtain 2010 ways of writing 2010 a sum of one or more positive integers in non-decreasing order such that the difference between the last term and the first term is at most 1.

20 Answer: 60.

Lets a be a positive integer such that $a + (a+1) + \ldots + (a + (n-1)) = 2010$.

Then $n\,a + \sum_{i=0}^{n-1} i = 2010$ so $n\,a + \dfrac{n(n-1)}{2} = 2010$ thus $\dfrac{n(2a + n - 1)}{2} = 2010$.

It gives $n(2a+n-1) = 4020 = 2^2\times3\times5\times67$.
Lets note that $2a+n-1 > n$ so $n < \sqrt{4020} = 2\sqrt{1005}$.
As $31^2 < 1005 < 32^2$, $n < 2\times32 = 64$.
Consequently n is a divisor of $2^2\times3\times5\times67$ such that $n < 64$.
So $n \in \{1; 2; 3; 4; 5; 6; 10; 12; 15; 20; 30; 60\}$.
Yet for $n = 60$, $2a+n-1 = \frac{4020}{60} = 67$ gives $a = 4$.
So 60 is a possible value and the least.

21 ◊ Answer: 2.

As $2!,\ldots, n!$ are all even numbers, we can assure that $1!+\ldots+n!$ is odd.
Therefore as the square of an number is even, if and only if the number is even, we can assure that m is odd.
There is no solution with m even.

As $3!, \ldots, n!$ are all multiples of 3, in the eucilidan division of $1!+\ldots+n!$ by 3, it remains the same remainder as $1!+2!$ divided by 3, it is 0.
Therefore m^2 is a multiple of 3, and as 3 is prime, m is also a multiple of 3.
Then m is an odd number, multiple of 3: $\{3; 9; 15; \ldots\}$.

As $4!, \ldots, n!$ are multiples of 4, the remainder of $1!+\ldots+n!$ divided by 4 is the same as $1!+2!+3! = 9$ divided by 4, it is 1.
Therefore $m^2 = 4q+1$ with $q \in \mathbb{Z}$.
This result isn't so intereting here, as the square of any even number has remainder 0 divided by 4, and the square of any odd number has remainder 1 divided by 4.
We just obtain the same result as with 2. m^2 is odd and so m also.

As $5!, \ldots, n!$ are all multiples of 5, the remainder of $1!+\ldots+n!$ divided by 5 is the same as $1!+2!+3!+4! = 33$ divided by 5, it is 3.
But there are no natural integer such that its square has remainder 3 divided by 5.
In fact, as any naturel integer can be written $10a+b$ with $a \in \mathbb{N}$ and $b \in \{0; 1; 2; \ldots; 9\}$, $m^2 = 10^2a^2 + 2\times10ab + b^2$ and because $10^2a^2 + 2\times10ab$ is divisble by 5, the remainder of m^2 divided by 5 is the remainder of b^2 divided by 5.
So we just have to examine the remainder of the squares of 0, 1, …,9 divided by 5.

integers	0	1	2	3	4	5	6	7	8	9
Remainder of the squares	0	1	4	4	1	0	1	4	4	1

So for all natural integers, the remainder of its square divided by 5 is 0, 1 or 4 not 3.
Consequently there is no positive integers n and m such that $n \geq 5$ and $1!+2!+3!+\ldots+n! = m^2$.

It remains to study the 4 first cases.
We have:

- $1! = 1^2$ so (n, m) is a pair solution.
- $1!+2! = 3$ and there is no integer such that $m^2 = 3$.
- $1!+2!+3! = 9 = 3^2$ so (3, 3) is also a solution.
- $1!+2!+3!+4! = 33$ and there is no integer such that $m^2 = 33$.

Finally, we have 2 solutions.

22 ◊◊ Answer: 65.

The triangles ΔABC and ΔABD are right-angled since C and D are points of the circle with diameter AB .
Therefore ΔADE is also rectangle.
Additionally, as $\angle DEA$ and $\angle BEC$ are opposite angles, they are congruent and so the triangles ΔADE and ΔBCE have the same angles: they are congruent.

We obtain $\dfrac{DE}{AE} = \dfrac{CE}{BE}$ so $x = \dfrac{169\times119}{BE}$.

Let's determine BE .

As ΔBEC is a right-angled triangle, $BE^2 = BC^2 + CE^2 = BC^2 + 119^2$.

As $\Delta\, ABC$ is a right-angled triangle, $BA^2 = BC^2 + (AE + EC)^2 = BC^2 + 288^2$.

As line BD is the angle bissector of the angle $\angle ABC$, we have $\dfrac{EC}{EA} = \dfrac{BC}{BA}$.

In fact, this classic resut comes reasoning with the areas of the triangles ΔBCE and ΔBEA .

As E is a point of the angle bissector of the angle $\angle ABC$, E is equidistant to the sides BA and BE .

So the two triangles have heights of common measure.

Let's note h the length of this heigths.

$A_{BAE} = \dfrac{AB\times h}{2}$ and $A_{BCE} = \dfrac{BC\times h}{2}$, so $h = \dfrac{2A_{BAE}}{AB} = \dfrac{2A_{BCE}}{AC}$ and finally $\dfrac{AC}{AB} = \dfrac{A_{BCE}}{A_{BAE}}$.

Now as B is a common vertex of the two triangles and as the points A, C, E are aligned, they also share the same height h' respectively to vertex B .

So far, $A_{BAE} = \dfrac{AE\times h'}{2}$ and $A_{BCE} = \dfrac{CE\times h'}{2}$ so $\dfrac{CE}{AE} = \dfrac{A_{BCE}}{A_{BAE}}$.

Finally we obtain $\dfrac{AC}{AB} = \dfrac{CE}{AE}$.

Thus, $BC = \dfrac{119}{169} BA$ and $BA^2 = \dfrac{288^2}{1-\frac{119^2}{169^2}} = \dfrac{169^2\,(169+119)^2}{(169-119)\,(169+119)} = 169^2\times\dfrac{288}{50} = \left(169\times\dfrac{12}{5}\right)^2$ so

$BA = \dfrac{12}{5}\times169.$

Folowing, $BE = \sqrt{\left(\dfrac{119}{169}\times BA\right)^2 + 119^2} = \sqrt{\dfrac{144}{25}\times119^2 + 119^2} = \sqrt{\dfrac{169}{25}\times119^2} = \dfrac{13}{5}\times119$.

Finally $x = \dfrac{169\times119}{\frac{13}{5}\times119} = \dfrac{5\times13^2}{13} = 5\times13 = 65$.

23 Answer: 72.

It is necessary that $0 \le n \le 190$ and $0 \le m \le 190$.

Moreover, if $(m; n)$ is an ordered pair of positive integers such that $m + n = 190$, $(n; m)$ also. Lets note $(n; m) \neq (m; n)$ if and only if $m \neq n$.

Then we can suppose $0 \le m \le n \le 190$ so that $0 \le m \le 95$ since $2m \le m + n \le 190$.

Lets $m \in [\![0; 95]\!]$.

We remark that $190 = 0 + 190 = 1 + 189 = 2 + 188 = \ldots 94 + 96 = 95 + 95$, it is $190 = m + (190 - m)$ for $m = 0$ to $m = 95$.

We count 95 ordered pairs $(m; n)$ such that $n \neq m$ and one ordered pair $(m; n)$ such that $m = n$.

We then count 191 ordered pairs $(m; n)$ such that $m + n = 190$.

Now we must deduct the pairs such that m and $n = 190 - m$ have a positive common divisor > 1 .

Lets d be a common divisor of m and n : $m + n = d\,q + d\,q' = d(q + q')$ so $d(q + q') = 190$: d is a divisor of 190.

Lets d be a common divisor of m and 190. So $n = 190 - m = d\,q - d\,q' = d(q - q')$ so d is a divisor of n .

Therefore m and n are relatively prime if and only if m and 190 are relatively prime.

As $190 = 2\times5\times19$, divisors of 190 are $\{1; 2; 5; 10; 19; 38; 95; 190\}$.

Then $m \in [\![0; 95]\!]$ is relatively prime with 190 if m is not divisible by a divisor of 190 .

The positive integer m, must not be divisible by 2, by 5, by 10, by 19, by 38 nor by 95.

There are 48 multiples of 2 belonging to $[\![0; 95]\!]$, 20 multiples of 5 belonging to $[\![0; 95]\!]$, 10 multiples of 10 belonging to $[\![0;95]\!]$, 6 multiples of 19 belonging to $[\![0;95]\!]$, 3 multiples of 38 belonging to $[\![0;95]\!]$, 2 multiples of 95 belonging to $[\![0;95]\!]$.

It leaves 96 integers m.

-48 (all even numbers including 0)

-9 (all multiples of 5 that are even)

-2 (multiples of 19 that are even)

-1 (for 95))

$= 36$

Finally there are 72 ordered pairs $(m; n)$ of positive integers such that $m + n = 190$.

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24 ◊◊◊◊ Answer: 32.

Find the least possible value of $f(x) = \dfrac{9}{1+\cos(2x)} + \dfrac{25}{1-\cos(2x)}$, where x ranges over all real numbers for which $f(x)$ is defined.

$f(x)$ is defined if $\cos(2x) \neq 1$ or $\cos(2x) \neq -1$ so $x \neq \dfrac{k\pi}{2}$, $k \in \mathbb{Z}$.

Moreover, the function f is π-periodic and even, the least possible value of $f(x)$ is reached over $]\,0;\dfrac{\pi}{2}[$.

In fact $f(x+\pi) = \dfrac{9}{1+\cos(2(x+\pi))} + \dfrac{25}{1-\cos(2(x+\pi))} =$ $\dfrac{9}{1+\cos(2x+2\pi)} + \dfrac{25}{1-\cos(2x+2\pi)} = \dfrac{9}{1+\cos(2x)} + \dfrac{25}{1-\cos(2x)} = f(x)$ and

$f(-x) = \dfrac{9}{1+\cos(2(-x))} + \dfrac{25}{1-\cos(2(-x))} = \dfrac{9}{1+\cos(2x)} + \dfrac{25}{1-\cos(2x)} = f(x)$ because the function cosine is 2π-periodic and even.

Let's detemrine its derivative.

For all $x \in \mathbb{R}\backslash\left\{\dfrac{k\pi}{2},\ k \in \mathbb{Z}\right\}$,

$$f'(x) = \frac{9\times 2\sin(2x)}{(1+\cos(2x))^2} + \frac{25\times(-2\sin(2x))}{(1-\cos(2x))^2} = \frac{2\sin(2x)}{(1+\cos(2x))^2(1-\cos(2x))^2}\left[9(1-\cos(2x))^2 - 25(1+\cos(2x))^2\right]$$

so $f'(x) = \dfrac{-8\sin(2x)}{(1+\cos(2x))^2(1-\cos(2x))^2} P(\cos(2x)$ avec $P(X) = 4X^2 + 17X + 4$.

For all $x \in]\,0;\dfrac{\pi}{2}[$, $2x \in]\,0;\pi[$ so $\sin(2x) > 0$.

Clearly, $(1+\cos(2x))^2(1-\cos(2x))^2 > 0$ for all $x \in]\,0;\dfrac{\pi}{2}[$.

Then $f'(x)$ has the opposite sign to $P(\cos(2x))$.

The polynomial $P(X)$ is a quadratic polynomial with discriminant $\Delta = 225 > 0$ so it has 2 roots $X_1 = \dfrac{-17-\sqrt{225}}{2\times 4} = -4$ and $X_2 = -\dfrac{1}{4}$.

We also obtain that $P(X) > 0$ for all $X > -\dfrac{1}{4}$ or $X < -4$ and negative otherwise.

As for all $x \in]\,0;\dfrac{\pi}{2}[$, $-1 < \cos(2x) < 1$, there is only one root left for $P(\cos(2x)) = 0$, it is $\cos(2x) = -\dfrac{1}{4}$.

Because the function $x \mapsto \cos(2x)$ is strictly decreasing over $]\,0;\dfrac{\pi}{2}[$, we can assure that:

- $\cos(2x) = -\frac{1}{4}$ has a unique solution over $]0; \frac{\pi}{2}[$, $x = \frac{1}{2}\arccos\left(-\frac{1}{4}\right) = \frac{1}{2}\cos^{-1}\left(-\frac{1}{4}\right)$
- $\cos(2x) \in]-1; -\frac{1}{4}]$ for all $x \in [\frac{1}{2}\arccos\left(-\frac{1}{4}\right); \frac{\pi}{2}[$ and $\cos(2x) > -\frac{1}{4}$ for all $x \in]0; \frac{1}{2}\arccos\left(-\frac{1}{4}\right)[$.

Consequently, $P(\cos(2x)) > 0$ for all $x \in]0; \frac{1}{2}\arccos\left(-\frac{1}{4}\right)[$ and $P(\cos(2x)) < 0$ for all $x \in]\frac{1}{2}\arccos\left(-\frac{1}{4}\right); \frac{\pi}{2}[$.

Finally, $f'(x) > 0$ for all $x \in]\frac{1}{2}\arccos\left(-\frac{1}{4}\right); \frac{\pi}{2}[$ and $f'(x) < 0$ for all $x \in]0; \frac{1}{2}\arccos\left(-\frac{1}{4}\right)[$.

The function f is then decreasing over $x \in]0; \frac{1}{2}\arccos\left(-\frac{1}{4}\right)[$ and increasing over $x \in]\frac{1}{2}\arccos\left(-\frac{1}{4}\right); \frac{\pi}{2}[$.

It admits a minimum at $x = \frac{1}{2}\arccos\left(-\frac{1}{4}\right)$.

This minimum is given by $f\left(\frac{1}{2}\arccos\left(-\frac{1}{4}\right)\right)$.

Let's remind that $\cos\left(2\left(\frac{1}{2}\arccos\left(-\frac{1}{4}\right)\right)\right) = -\frac{1}{4}$ so $f\left(\frac{1}{2}\arccos\left(-\frac{1}{4}\right)\right) = \frac{9}{1-\frac{1}{4}} + \frac{25}{1+\frac{1}{4}} = 12 + 20 = 32$.

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25 Answer: 2002.

As there must be at least one blue ball between any two red balls, and as the 5 red balls are arranged on the straight line, 4 blue balls are automatically arranged the red balls being arranged.
Then we can deduce that the problem is equivalent to count the number of combination of 9 blue balls and 5 red balls, so equivalent to the number of ways of arranging 5 red balls in $9 + 5 = 14$ places.
Therefore we have $\frac{14\times13\times12\times11\times10}{1\times2\times3\times4\times5} = 2002$ ways of arranging 13 identical blue balls and 5 identical red balls on a straight line such that between any 2 red balls there is at least 1 blue ball.

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26 ◊◊ Answer: 49.

As $49\times2010 < 100\,000 < 50\times2010$, we have $k = 49$.
In fact for all subset A such that $\lfloor A\rfloor = 2010$, the subset A contains 2010 distinct numbers and so there are at least 2 distinct numbers a and b such that $|a-b| \le 49$.
Let's $A = \{a_1; a_2; \ldots; a_{2010}\}$ with $a_1 < a_2 < \ldots < a_{2010}$.
If for all i and j, $|a_i - a_j| > 49$, so $a_{i+1} \ge a_i + 50$ and then as $a_1 > 1$, $a_{2010} \ge 1 + 2009\times50 > 100\,000$.

Remark:

Using the partition $\{1; \ldots; 50\} \bigcup \{51; \ldots; 100\} \bigcup \ldots \bigcup \{99\,951, 100\,000\} = \bigcup_{i=0}^{1999}\{1 + 50i; 2 + 50i; \ldots; 50 + 50i\}$ (2000 subsets of length $49 = 50 - 1$), of $\{1; \ldots; 100\,000\}$, for any subset A such that $\lfloor A\rfloor = 2010 > 2000$, by the Pigeonhole Principle, there exist at least one subset $\{1 + 50i; 2 + 50i; \ldots; 50 + 50i\}$ that contains at least 2 elements of A.
Therefore there exists at least two elements a and b such that $|a - b| \le 49$.

Let's built a subset such that there ain't any distinct numbers a and b such that $|a - b| < 49$.
We part $\{1; \ldots; 100\,000\}$ in 2000 subsets: $\{1; \ldots; 50\} \bigcup \{51; \ldots; 100\} \bigcup \ldots \bigcup \{99\,951, 100\,000\}$.
Let's define $A = \{1; 50; 99; \ldots; 98\,442\} = \{49j + 1; j = 0, 1, 2, \ldots 2009\}$.
We have $\lfloor A\rfloor = 2010$ and as $(49(j+1) + 1) - (49j + 1) = 49$, then for two consecutive numbers a and b of A, $|a - b| = 49$ and for two distinct numbers of A, $|a - b| \ge 49$.

Consequently, 49 is the least possible value of k such that any subset A of S with $\lfloor A\rfloor = 2010$ contains two distinct numbers a and b with $|a - b| \le k$.

27 Answer: 112.

As we are only allowed to travel east or north, from A to B, in order to avoid the 4 points ×, we go through points C, D or E as shown on the figure below, and then from C, D or E to point B .

From A to C ,there is only one way to go: 4 steps East.

From C to B , there are 2 steps East to do and 5 steps North, so there are $\binom{7}{2} = 21$ different ways.

From A to D , there are $\binom{4}{1} = 4$ ways, and from D to B, there are $\binom{7}{2} = 21$ different ways.

So from A to B going through D, there are $4 \times 21 = 84$ different ways.

From A to E, there is only way to go, but from E to B there are $\binom{7}{1} = 7$ different ways.

Finally there are $21 + 84 + 7 = 112$ different ways to go from A to B with the conditions given.

28 Answer: 60.

Let's note O_1 and O_2 the centres respectively of C_1 and C_2.
As line PM is tangent to circle C_2, the triangle O_2PM is right-angled at M .

Therefore, Pythagoreas' theorem gives: $O_2P^2 = MP^2 + MO_2^2 = 8^2 + \left(\sqrt{20}\right)^2 = 84$.

In triangle APO_2, using the law of cosines, we thus obtain $O_2P^2 = AO_2^2 + AP^2 - 2\,AP \times AO_2 \cos(x)$ so $84 = 8^2 + AP^2 - 16\,AP\cos(x)$ and finally $AP^2 - 16\,AP\cos(x) = 20$.

In triangle APO_1, we obtain $O_1P^2 = AP^2 + AO_1^2 - 2\,AP \times AO_1 \cos(x)$ so $AP^2 - 20\,AP\cos(x) = 0$ as $O_1P = AO_1$.

Finally, we have as $AP \neq 0$, $AP = 20\cos(x)$ so $400\cos^2(x) - 16\,(20\cos(x))\cos(x) = 20$.

We deduce $\cos^2(x) = \frac{20}{80} = \frac{1}{4}$ and then $\cos(x) = \frac{1}{2}$.
Consequently, $x° = 60°$.

29 Answer: 6.

Let a, b and c be integers with $a > b > c > 0$. If b and c are relatively prime, $b + c$ is a multiple of a, and $a + c$ is a multiple of b, determine the value of abc .
As $a > b$ and $a > c$ then $2a > b + c$.
Therefore as $b + c$ is a strictly positive integer multiple of a, it falls $b + c = a$. This is the only multiple of a, strictly positive and strictly inferior to $2a$.
Then $a + c = b + c + c = b + 2c$.
As $a + c$ is a multiple of b, $a + c - b$ also and so $2c$ is a multiple of b .
But as b and c are relatively prime, c is not a multiple of b .
We thus obtain that 2 is a multiple of b . So $b = 1$ or $b = 2$.
As $0 < c < b$ then $b > 1$ so $b = 2$.
It follows $c = 1$ and $a = 3$.
Consequently, $abc = 6$.

30 ◊ Answer: 680.

Find the number of subsets $\{a, b, c\}$ of $\{1, 2, 3, 4, \ldots, 20\}$ such that $a < b - 1 < c - 3$.
As $a < b - 1$ then $b \geq a + 2$ and as $b - 1 < c - 3$, thus $c \geq b + 3$.
Then as $c \leq 20$, $b \leq 17$ and $a \leq 15$.
The number of subsets $\{a, b, c\}$ of $\{1, 2, 3, 4, \ldots, 20\}$ such that $a < b - 1 < c - 3$ is the number of subsets $\{a, b, c\}$ such that when a describes $\{1; \ldots; 15\}$, b describes $\{a + 2; \ldots; 17\}$ and c describes $\{b + 3; \ldots; 20\}$.

With $a = 1$, we can have:

- $b = 3$ so $c \in \{6; \ldots; 20\}$: 15 choices.
- $b = 4$ so $c \in \{7; \ldots 20\}$: 14 choices.

$\ldots$

- $b = 17$ so $c = 20$: 1 choice.

There are $1 + 2 + \ldots + 15 = \dfrac{16\times 15}{2} = 120$ possibilities with $a = 1$.

With $a = 2$, we have:

- $b = 4$ so $c \in \{7; \ldots; 20\}$: 14 choices.
- $b = 5$ so $c \in \{8; \ldots; 20\}$: 13 choices.

$\ldots$

- $b = 17$ so $c = 20$: 1 choice.

There are $1 + 2 + \ldots + 14 = 105$ possibilities with $a = 2$.

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With $a = 15$, $b = 17$ and $c = 20$: there is 1 possibility with $a = 15$.

Finally we count $1 + (1 + 2) + (1 + 2 + 3) + \ldots + (1 + 2 + 3 + \ldots + 15) = 15\times 1 + 14\times 2 + \ldots + 1\times 15 = 680$.

Remark:

We also can associate each subset $\{a, b, c\}$ such that $a < b - 1 < c - 3$ to a subset $\{\alpha, \beta, \gamma\}$ such that $\alpha, \beta, \gamma \in \{1; 2; \ldots; 17\}$.

There $\binom{17}{3} = \dfrac{17\times 16\times 15}{1\times 2\times 3} = 680$ possibilities.

31 ◊ Answer: 2780.

For n integer between 1 and 99 999, $f(n)$ takes the values: 0, 1, 2, 3 and 4.

So $$M = \sum_{f(i)=0} 0\times 2^0 + \sum_{f(i)=1} 1\times 2^1 + \sum_{f(i)=2} 2\times 2^2 + \sum_{f(i)=3} 3\times 2^3 + \sum_{f(i)=4} 4\times 2^4$$ so

$$M = \begin{array}{r} 2\times(\text{number of numbers with one } 0) \\ +8\times(\text{number of numbers with two } 0's) \\ +24\times(\text{number of numbers with three } 0's) \\ +64\times(\text{number of numbers with four } 0's) \end{array}$$

We then need to count how numbers of each type.

The numbers with one 0 can be written:

- $abcd0$, $abc0d$, $ab0cd$, $a0bcd$ with $a, b, c, d \in \{1; 2; \ldots; 9\}$.
  There are 4×9^4 numbers of this type.
- $abc0$, $ab0c$, $a0bc$ with $a, b, c \in \{1; 2; \ldots; 9\}$.
  There are 3×9^3 numbers of this type.
- $ab0$, $a0b$ with $a, b \in \{1; 2; \ldots; 9\}$.
  There are 2×9^2 numbers of this type.
- $a0$ with $a \in \{1; 2; \ldots; 9\}$.
  there are 9 numbers of this type.

Finally there are $4\times 9^4 + 3\times 9^3 + 2\times 9^2 + 9 = 28\,602$ numbers between 1 and 99 999 with one 0.

The numbers with two 0's can be written:

- $abc00$, $ab0c0$, $a0bc0$, $ab00c$, $a00bc$ or $a0b0c$ with $a, b, c \in \{1; 2; \ldots; 9\}$.
  There are 6×9^3numbers of this type.
- $ab00$, $a0b0$ or $a00b$ with $a, b \in \{1; 2; \ldots; 9\}$.
  There are 3×9^2 numbers of this type.
- $a00$ with $a \in \{1; 2; \ldots; 9\}$.
  There are 9 numbers of this type.

Finally there are $6\times 9^3 + 3\times 9^2 + 9 = 4626$ numbers between 1 and 99 999 with two 0's.

The numbers with three 0's can be written:

- $a\,b\,000$, $a\,0\,b\,00$, $a\,00\,b\,0$ or $a\,000\,b$ with $a,\ b \in \{1; 2; \ldots; 9\}$.
  There are 4×9^2 numbers of this type.
- $a\,000$ with $a \in \{1; 2; \ldots; 9\}$.
  There are 9 numbers of this type.

Finally there are $4\times 9^2 + 9 = 333$ numbers between 1 and 99 999 with three 0's.

The numbers with four 0's are $a\,0000$ with $a \in \{1; 2; \ldots; 9\}$. there are 9 numbers with four 0's.

We deduce:
$M = 2\times 28\,602 + 8\times 4626 + 24\times 333 + 64\times 9 = 102\,780$.
thus $M - 100\,000 = 2780$.

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32 ◊◊ Answer: 5.

Determine the odd prime number p such that the sum of digits of the number $p^4 - 5\,p^2 + 13$ is the smallest possible.

For $p = 3$, $n = 3^4 - 5\times 3^2 + 13 = 81 - 45 + 13 = 49$ so the sum of the digits is 13.

For $p = 5$, $n = 5^4 - 5\times 5^2 + 13 = 625 - 125 + 13 = 513$ so the sum of the digits is 9.

For $p = 7$, $n = 7^4 - 5\times 7^2 + 13 = 2401 - 245 + 13 = 2169$ so the sum is greater than 9.

For $p = 11$, $n = 11^4 - 5\times 11^2 + 13$.
Let's note that therefore the last digit of n is given by $\sharp 1 - 5 + 3 = 6 + 3 = 9$ so the sum of the digits of n is at least 9.

Next for all prime > 11, the last digit of n is 9.
In fact, the last digit of p^4 is given by the last digit of 1^4, 3^4, 7^4 or 9^4 as an odd prime ends by 1, 3, 7 or 9.

As $1^4 = 1$, $3^4 = 81$, $7^4 = 2401$, and $9^4 = 81^2 = \sharp 1$ so the last digit of p^4 is 1.

Then the last digit of p^2 is the one of 1^2, 3^2, 7^2 or 9^2 so it is 1 or 9.

Therefore the last digit of $5\times p^2$ is 5.
Finally the last digit of n is given by $\sharp 1 - 5 + 3 = 9$.

Consequently, the sum of the digits of n is at least 9.
9 is the smallest possible and is rached for $p = 5$.

Remark:
$p^4 - 5\,p^2 + 13 = p^4 - 5\,p^2 + 4 + 9 = (p - 2)\,(p - 1)\,(p + 1)\,(p + 2) + 9$.
We can notice that $(p - 1)\,(p + 1) = (p^2 - 1)$, $(p - 2)\,(p + 2) = p^2 - 4$ and $X^2 - 5\,X + 4 = (X - 1)\,(X - 4)$, hence the factorisation of $p^4 - 5\,p^2 + 4$.
For $p > 5$, $p - 2,\ \ p - 1,\ \ p,\ \ p + 1,\ \ p + 2$ are 5 consecutive integers.
As p is prime, one of $p - 2,\ \ p - 1,\ \ p + 1,\ \ p + 2$ is multiple of 5 and 2 are even.
So $(p - 2)\,(p - 1)\,(p + 1)\,(p + 2)$ is a multiple of 10.
Consequently the last digit of n is 9.

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33 Answer: 360.

As $B\,C = 3\,A\,D = 3\,C\,E$, we deduce $A\,D = C\,E$ and denote a this length.
Let's note h the height in triangle GCE relative to edge G and h' the height of trapezium $ABCD$.

We thus have $A_{GCE} = \dfrac{a\,h}{2} = 15$ and $k = \dfrac{4\,a\times h'}{2} = 2\,a\,h'$.

Let H be the point of $D\,C$ such that $F\,H\,//\,A\,D$.

As $A\,D\,//\,F\,H\,//\,B\,C$ and as F is the midpoint of AB, point H is the midpoint of CD, $F\,H = \dfrac{1}{2}\,(A\,D + B\,C) = 2\,a$ and the distance between lines AD and FH is equal to the distance between lines FH and BC, equal to half the height h' of the trapezium.

As $F H \,//\, C E$ and the angles $\angle F G H$ and $\angle C G H$ are opposite angles so congruent, the triangles *FGH* and *EGC* are similar.
We deduce that the triangle *FGH* is the enlargment in the ratio 2 of the triangle *EGC*.
Thus its height issued at *G* in triangle *EGC* is 2 times the corresponding height at *G* in triangle *EGC* as edge *G* is common.
Consequently the distance between lines *FH* and *BC* is equal to $3\,h$.
And thus the distance between lines *BC* and *AD* is $6\,h$: $h' = 6\,h$.

We obtain $\dfrac{a \times \frac{h'}{6}}{2} = 15$ so $a\,h' = 12 \times 15 = 180$.
Finally $k = 2\,a\,h' = 2 \times 180 = 360$.

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34 ◊◊ Answer: 75.

As $P(1) = 25$, then $a_0 + a_1 + \ldots + a_n = 25$.
As the coefficients $a_0, a_1, a_2, \ldots, a_n$ are non-negative integers, $0 \le a_i \le 25$ for $i = 1$ to n.

As $P(27) = 1\,771\,769$, then $a_0 + 27\,a_1 + 27^2\,a_2 + \ldots + 27^n\,a_n = 1\,771\,769$.
As $0 \le a_i \le 25 < 27$ for $i = 1$ to n, we have $1\,771\,769 = \overline{a_n \ldots a_2\,a_1\,a_0}^{27}$, it is to say that $a_n \ldots a_2\,a_1\,a_0$ is the writing of $1\,771\,769$ in base 27.
Yet $1\,771\,769 = 65\,621 \times 27 + 2 = (2430 \times 27 + 11) \times 27 + 2 = (90 \times 27 \times 27 + 11) \times 27 + 2$.
So $1\,771\,769 = (3 \times 27 + 9) \times 27^3 + 11 \times 27 + 2 = 3 \times 27^4 + 9 \times 27^3 + 11 \times 27 + 2$.
It follows that $n = 4$, $a_0 = 2$, $a_1 = 11$, $a_2 = 0$, $a_3 = 9$ and $a_4 = 3$.
Hence $P(x) = 2 + 11\,x + 9\,x^3 + 3\,x^4$.

$a_0 + 2\,a_1 + 3\,a_3 + 4\,a_3 + 5\,a_4 = 2 + 2 \times 11 + 3 \times 0 + 4 \times 9 + 5 \times 3 = 75$.

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35 ◊◊◊ Answer: 12.

Let three circles $\Gamma_1, \Gamma_2, \Gamma_3$ with centres A_1, A_2, A_3 and radii r_1, r_2, r_3respectively be mutually tangent to each other externally.
Suppose that the tangent to the circumcircle of the triangle $A_1\,A_2\,A_3$ at A_3 and the two external common tnagents of $\Gamma_1$and $\Gamma_2$meet a common point *P*, as shown on the figure below.
Given that $r_1 = 18$ cm, $r_2 = 8$ cm and $r_3 = k$ cm, find the value of k.
First, as the two external common tangents of Γ_1 and Γ_2 meet in *P*, by symmetry, *P*, A_1 and A_2 are aligned.
Moreover, we can define two right-angled triangles $P\,A_1\,E_1$ and $P\,A_2\,E_2$ where E_1 and E_2 are the points of tangency of one tangent with respectively circle C_1 and circle C_2.

These two triangles are similar and so $\dfrac{P\,A_2}{P\,A_1} = \dfrac{r_2}{r_1}$.

Second, the power of point *P* relatively to the circumcircle of triangle $A_1\,A_2\,A_3$ is known independant of the chosen point on this circumcircle.
We so have $P\,A_3{}^2 = P\,A_1 \times P\,A_2 = P\,O^2 - r^2$ with *O* centre of the circumcircle and *r* its radius.

Therefore $\dfrac{P\,A_3}{P\,A_2} = \sqrt{\dfrac{P\,A_1}{P\,A_2}} = \sqrt{\dfrac{r_1}{r_2}}$.

Third the law of sines in triangle $P\,A_2\,A_3$ gives $\dfrac{P\,A_3}{\sin(\angle A_3\,A_2\,P)} = \dfrac{P\,A_2}{\sin(\angle A_2\,A_3\,P)}$ so $\dfrac{P\,A_3}{P\,A_2} = \dfrac{\sin(\angle A_3\,A_2\,P)}{\sin(\angle A_2\,A_3\,P)}$.

The law of sines in triangle $A_1\,A_2\,A_3$ gives $\dfrac{\sin(\angle A_1\,A_2\,A_3)}{\sin(\angle A_2\,A_1\,A_3)} = \dfrac{r_1 + r_3}{r_2 + r_3}$.

As the angles $\angle A_1\,A_2\,A_3$ and $\angle A_3\,A_2\,P$ are complementary angles, then $\sin(\angle A_1\,A_2\,A_3) = \sin(\angle A_3\,A_2\,P)$.

Then let's show that $\angle A_2\,A_1\,A_3 = \angle A_2\,A_3\,P$.
Let *O* be the centre of the circumcircle of triangle $A_1\,A_2\,A_3$.

Then $2\angle A_2A_1A_3 = \angle A_2OA_3$ using the inscribded angle theorem.
As $OA_2 = OA_2$, $\angle OA_2A_3 = \angle OA_3A_2$ and then $\angle A_2OA_3 + 2\angle OA_3A_2 = \pi$.
Moreover, as line PA_3 is tangent to the circumcircle with center O, $\angle OA_3A_2 + \angle PA_3A_2 = \dfrac{\pi}{2}$.
We deduce $\angle OA_3A_2 = \frac{\pi}{2} - \angle PA_3A_2$ so substituing in the first equality, we obtain $\angle A_2OA_3 = \pi - 2\left(\frac{\pi}{2} - \angle PA_3A_2\right) = 2\angle PA_3A_2$ and finally $\angle A_2A_1A_3 = \angle A_2A_3P$.
Thus $\sin(\angle A_2A_1A_3) = \sin(\angle A_2A_3P)$.

So we obtain $\dfrac{PA_3}{PA_2} = \dfrac{r_1 + r_3}{r_2 + r_3}$.

Finally, $\sqrt{\dfrac{r_1}{r_2}} = \dfrac{r_1 + r_3}{r_2 + r_3}$.

So $(r_2 + r_3)\sqrt{r_1} = (r_1 + r_3)\sqrt{r_2}$ then $r_3\left(\sqrt{r_1} - \sqrt{r_2}\right) = r_1\sqrt{r_2} - r_2\sqrt{r_1} = \sqrt{r_1r_2}\left(\sqrt{r_1} - \sqrt{r_2}\right)$: $r_3 = \sqrt{r_1r_2}$.

Consequently, $r_3 = \sqrt{18\times 8} = \sqrt{144} = 12$.